Technical Note

Generalized heat conduction model in moving media emanating from Boltzmann Transport Equation

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ABSTRACT

In this work, we start with several concerns regarding Galilean variance, pointed out by Christov and Jordan (2005), of the Cattaneo-Vernotte heat conduction in moving media. We then describe a generalized heat transport model in moving media and its underlying theory emanating from the Boltzmann Transport Equation, which achieves the Galilean invariance in all the inertial frameworks. The resulting model recovers heat transport characteristics of different scales with respect to both space (from ballistic to diffusive limits) and time (from finite to infinite heat propagation speeds).

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1. Introduction

Fourier's law has been widely exploited to describe heat conduction and mass transport phenomenon for most engineering applications. However, Fourier's law is acknowledged to be local in time and predicts thermal wave propagation with an infinite speed [1], which is unrealistic. To address this 'paradox of heat conduction', various modifications of the established Fourier's law, such as Cattaneo-Vernotte [2,3], have been proposed. Later, a Dual-Phase-Lag (DPL) [1,4] model was proposed as a generalized heat conduction model by adding two phase lags into the Fourier's Law; however, this DPL model is physically inaccurate as noted in the literature [6]. Alternately, the C- and F-heat conduction model proposed by Tamma et al. [5,6] describes the evolution of heat transport characteristics of different scales with respect to both space (from ballistic to diffusive limits) and time (from finite to infinite heat propagation speeds).

Regarding the heat conduction in moving media [7], the material derivative with respect to time has been exploited in the balance law for the internal (heat) energy, which is expressed as

$$\rho c_p \frac{\partial T}{\partial t} + \boldsymbol{V} \cdot \nabla T + \nabla \cdot \boldsymbol{q} = 0 \tag{1}$$

where T and $\boldsymbol{q}$ represent the temperature and the thermal flux vector, respectively; c_p and ρ represent the specific heat and the

density of material, respectively; $\boldsymbol{V}$ is the velocity of moving media. The $\boldsymbol{q}$ of Fourier's law and its modification, Cattaneo-Vernotte model, are as follows:

1. The Fourier Model

$$\boldsymbol{q} = -k\nabla T \tag{2}$$

where k is the conductivity.

2. The Cattaneo Model

$$\tau \frac{D\boldsymbol{q}}{Dt} := \tau \left(\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{q} \right) - \boldsymbol{q} = -k\nabla T \tag{3}$$

where τ is the relaxation time. Coupled with the balance equation in moving media, Eqs. (1) and (3) end up with a diffusive-hyperbolic heat conduction model. In addition, the speed of second sound is given as

$$c = \sqrt{\frac{k}{\rho c_p \tau}} \tag{4}$$

One may note that the partial time derivative in the original Cattaneo-type thermal flux is replaced by a material derivative in Eq. (3). This is in order to circumvent the paradox of the second sound inconsistency in different inertial frameworks when the Cattaneo-type heat conduction model is exploited to depict the heat transfer in moving media. Specifically, the paradox is that the thermal waves of finite speed $c = \sqrt{\frac{k}{\rho c_p \tau}}$ in the usual form, which involves a partial time derivative, of Cattaneo-type heat con-

duction model is different from the one in the moving media from the resting body, which violates the Galileo's principle of relativity. This remarkable replacement was firstly introduced by [7], which has made a significant contribution with respect to the field of heat transport in the moving media. The associated model was proved to preserve the Galilean invariance in all the inertial frameworks. More details can be found in [7].

Inspired by [7], the present work is designed as an extension of the previous studies and devotes to achieve: (a) interpreting the underlying physics of the heat conduction in moving media from the perspective of Boltzmann Transport Equation. Apart from the mathematical presence of the replacement proposed previously, physical interpretations behind this simple replacement is worth being noted; (b) exploring the Galilean invariance of the Fourier-, Cattaneo-, and Jeffreys-type heat transport processes for moving media; and (c) analyzing the limitations of the moving media speed.

2. Theory

2.1. The BTE in the material framework

Consider a very small region of space, Ω, namely system *I*, centered at the position $\boldsymbol{x}$ and the heat carrier has the velocity $\boldsymbol{v}$ at an instant of time. The BTE of the probability, $f(\boldsymbol{x}, \boldsymbol{v}, t)$, that a particle occupying a very small given region of space is given as:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{5}$$

where $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is the nonequilibrium thermodynamic distribution function, $\boldsymbol{v}(\omega)$ is the heat carrier velocity and is assumed to be constant over a large frequency range such that $\frac{\partial f}{\partial \boldsymbol{v}}$ is omitted, $\boldsymbol{a}$ is the heat carrier acceleration due to the external force field.

Suppose system *II* has a relative velocity $\boldsymbol{V}$ with respect to system *I* described by Eq. (5), such that the transformation law between system *I* to *II* are as follows:

$$s = t, \quad \boldsymbol{X} = \boldsymbol{r} - \boldsymbol{V}s, \quad f(\boldsymbol{X}, T, E(\omega)) = f(\boldsymbol{x}, T, E(\omega))$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \boldsymbol{V}\frac{\partial}{\partial \boldsymbol{X}}, \quad \frac{\partial}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{X}} \tag{6}$$

where $s, \boldsymbol{X}$ are the time dimension and space vector in the system *II*, respectively. Note that the particle distribution f and t are assumed to be invariant in different frames. In such a way, the BTE in system *I* can be written in the sense of moving media, system *II*, as follows:

$$\frac{\partial f}{\partial s} + \boldsymbol{V} \cdot \frac{\partial f}{\partial \boldsymbol{X}} + \boldsymbol{v}' \cdot \frac{\partial f}{\partial \boldsymbol{X}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{7}$$

where $\boldsymbol{v}'$ is the velocity of heat carrier in system *II*. One may note that the right hand collision term is preserved the same as Eq. (5) under the assumption that the particle distributions of both systems are invariant.

Suppose that both Cattaneo-type slow processes/low frequency (C-processes) and Fourier-type fast processes/high frequency (F-processes) coexist concurrently in the heat conduction process, the total heat flux simultaneously accounts for low and high energy processes as

$$\boldsymbol{q} = \int_0^{\omega_D} \boldsymbol{v}' f(\boldsymbol{x}, T, \omega) \hbar\omega D(\omega) d\omega$$
$$= \int_0^{\omega_T} \boldsymbol{v}' f(\boldsymbol{x}, T, \omega) \hbar\omega D(\omega) d\omega \tag{8}$$
$$+ \int_{\omega_T}^{\omega_D} \boldsymbol{v}' f(\boldsymbol{x}, T, \omega) \hbar\omega D(\omega) d\omega = \boldsymbol{q}_C + \boldsymbol{q}_F$$

where $\hbar$ is Planck's number divided by 2π meaning $\hbar\omega$ is the energy, and $D(\omega)$ is the density of states. The assumption made here

is that the integral up to a threshold frequency ω_T associates with the slow process ($\boldsymbol{q}_C$) and that the integral from the threshold to the Debye frequency, ω_D, associates the fast process ($\boldsymbol{q}_F$). Also, the velocity of heat carrier, $\boldsymbol{v}'$, in Eq. (8), is the *local* heat carrier speed in each inertial system. The assumption of the coexistence of the fast and the slow heat conduction processes was proposed by Tamma et al. [5]. It provides the possibility of bridging the heat transport systems with multi-scale in both time (infinite wave speed to finite wave speed) and space (ballistic to diffusion).

Multiplying the transient BTE, Eq. (7), by $\boldsymbol{v}'\psi$, where $\psi = \hbar\omega D(\omega)$, and integrating over the entire frequency ranges (0 to ω_D) yields

$$\int_0^{\omega_T} \frac{\partial f}{\partial s} \boldsymbol{v}'\psi d\omega + \int_{\omega_T}^{\omega_D} \frac{\partial f}{\partial s} \boldsymbol{v}'\psi d\omega + \int_0^{\omega_T} \boldsymbol{V} \cdot \nabla_X f \boldsymbol{v}'\psi d\omega$$
$$+ \int_{\omega_T}^{\omega_D} \boldsymbol{V} \cdot \nabla_X f \boldsymbol{v}'\psi d\omega + \int_0^{\omega_T} \boldsymbol{v}' \cdot \nabla_X f \boldsymbol{v}'\psi d\omega + \int_{\omega_T}^{\omega_D} \boldsymbol{v}' \cdot \nabla_X f \boldsymbol{v}'\psi d\omega$$
$$= \int_0^{\omega_D} \left(\frac{\partial f}{\partial s}\right)_{coll} \boldsymbol{v}'\psi d\omega \tag{9}$$

where ∇_X is the gradient operator with respect to the space vector $\boldsymbol{X}$ in system *II*.

As a matter of fact, the distribution function, f, at high frequencies is nearly constant over time such that its time rate of change, $\int_{\omega_T}^{\omega_D} \frac{\partial f}{\partial s} \boldsymbol{v}'\psi d\omega$, can be neglected. This leads to the pure Fourier's model, whereas, the change of f mainly occurs within the low frequency domain. Before carrying on, the following approximations and definitions may be required in the following total flux splitting process.

1. Approximation of $\left(\frac{\partial f}{\partial s}\right)_{coll}$
   The Bhatnagar-Gross-Krook approximation is exploited in this work as follows:

   $$\left(\frac{\partial f}{\partial s}\right)_{coll} = \frac{f_0 - f}{\tau} \tag{10}$$

   where f_0 is the thermodynamic distribution at equilibrium and τ is the relaxation time for returning from the state f to equilibrium f_0.

2. Approximation of $\frac{\partial f}{\partial x}$
   According to the local thermal dynamic equilibrium, $\frac{\partial f}{\partial x}$ can be approximated as:

   $$\frac{\partial f}{\partial x} \cong \frac{\partial f_0}{\partial x} = \frac{df_0}{dT}\frac{dT}{dx} \tag{11}$$

3. Definition of specific heat and thermal conductivity
   Based on the kinetic theory, the specific heat and the thermal conductivity can be given under the presence of the temperature gradient as follows:

   $$C(T) = \int_0^{\omega_D} \frac{df_0}{dT} \psi d\omega;$$
   $$K(T) = \int_0^{\omega_D} \tau v'^2 \frac{df_0}{dT} \psi d\omega = \frac{1}{3} C\tau v^2 = \frac{1}{3} C\tau(v_C^2 + v_F^2) = K_C + K_F \tag{12}$$

   where $K(T)$ is the total thermal conductivity, $C(T)$ is the total specific heat per unit volume, v is the average speed of the heat carriers. Herein, two average speeds of heat carriers in different ranges are introduced for the forthcoming derivation. Specifically, v_C and v_F represent the average speeds of heat carriers in slow and fast heat processes, respectively, and are given by

   $$v_C = v\sqrt{1 - F_T}; \quad v_F = v\sqrt{F_T} \tag{13}$$