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Heat transfer in generalized vortex flow over a permeable surface

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ABSTRACT

Heat transfer in the thermal boundary layer beneath a generalized vortex flow has been considered. The steadily revolving flow is allowed to vary with the distance r from the symmetry axis as r^m. The governing equations for heat and momentum transport transformed exactly to a coupled set of ordinary differential equations by means of a tailor-made similarity transformation. Some different flow situations in presence of suction have been considered, including solid-body rotation $(m = +1)$ and a potential vortex $(m = -1)$. The thermal boundary layer was observed to thicken monotonically with decreasing m-values, accompanied by a reduction of the heat transfer rate through the planar surface above which the flow revolves. These findings were explained as the combined influence of two different effects, namely: (i) a variation of the effective Prandtl number $(m + 3)Pr/2$ that directly affected the thermal diffusion, whereas (ii) an indirect variation of the axial velocity component affected the thermal convection.

1. Introduction

Bödewadt [1] considered the steadily revolving flow of a viscous fluid above an impermeable solid surface. The fluid far above the planar surface was assumed to be in a state of rigid-body rotation so that the tangential velocity component v increased linearly with the distance r from the axis of rotation. The presence of viscous shear stresses inevitably slowed down the revolving fluid motion in the viscous boundary layer adjacent to the surface. A resulting imbalance between the radial pressure gradient and the centrifugal force gave rise to a velocity component directed towards the axis of rotation. Finally, to secure mass conservation, an axial flow away from the surface arose and a three-dimensional boundary layer flow was established. The effects of alternative boundary conditions at the planar surface have been examined by Nath and Venkatachala [2], Sahoo et al. [3] and Turkyilmazoglu [4]. They considered suction, partial slip, and stretching, respectively.

The classical Bödewadt flow problem was later generalized by King and Lewellen [5] who assumed the tangential velocity to vary as $v \sim r^m$ where m is dimensionless constant. They were partly concerned with the effect of the parameter m and partly with the effect of a magnetic body-force term on the fluid motion. Their study was extended by Venkatachala and Nath [6] to also include effects of suction through the surface. The generalized Bödewadt flow was also considered as a part of an extensive paper by Kuo [7] focused on tornado-like vortices. In addition to the conventional no-slip conditions at the solid surface, Kuo [7] also allowed for partial surface slip. This was referred to as a geophysical boundary condition.

It was suggested by Moore [8] that the generalized Bödewadt flow does not admit similarity solutions for $m = -1$, i.e. when the revolving flow behaves as a potential vortex. The non-existence of similarity solutions was later proved by King and Lewellen [5]. However, as demonstrated by Nanbu [9] and Venkatachala and Nath [6], similarity solutions do exist in the presence of suction through the surface.

The aim of the present paper is to investigate for the first time the heat transfer between a generalized Bödewadt flow and the planar surface above which the fluid revolves. We recently demonstrated that realistic similarity solutions of the thermal energy equation do not exist for the classical Bödewadt flow above an impermeable surface [10]. In presence of sufficent suction, however, the thermal boundary layer problem allowed for realistic similarity solutions. In order to extend the thermal analysis to generalized Bödewadt flows, accurate solutions of the three-componential velocity field are required. To this end we revisit the work by King and Lewellen [5] and Kuo [7] to first obtain the revolving flow field over an impermeable surface. Subsequently, distributed suction will be introduced, similarly as in the study by Venkatachala and Nath [6]. Finally, similarity solutions of the thermal energy equation will be provided for some different revolving flows, including solid-body rotation $(m = +1)$ and potential vortex flow $(m = -1)$, and for some different Prandtl numbers.

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2. Problem formulation and solution approach

2.1. Mathematical model equations

Let us consider the steadily revolving flow of a viscous fluid above a planar surface. In cylindrical polar coordinates (r, θ, z) the governing mass conservation, momentum and thermal energy equations become:

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{1}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{2}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{3}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right], \tag{4}$$

$$\rho C_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right), \tag{5}$$

where (u, v, w) are the velocity components of the fluid in the radial, tangential and axial directions, respectively, and T is the temperature. Here, we have assumed rotational symmetry about the vertical z-axis, *i.e.* $\partial/\partial\theta = 0$. The kinematic viscosity of the fluid is ν. C_p is the specific heat at constant pressure of the fluid. k is the thermal conductivity of the fluid.

This set of coupled partial differential equations (PDEs) are the same as those governing the heat and momentum transport in the von Karman flow driven by a steadily rotating disk and the Bödewadt flow caused by revolving fluid in solid-body rotation above a planar surface. In this paper, however, we are concerned with a generalized vortex flow which includes the classical Bödewadt flow [1] only as a special case. The boundary conditions are therefore given as

$$\begin{aligned} &u = 0, v = 0, \quad w = Av_0\sqrt{\frac{\nu}{v_0 r_0}}\left[\frac{r}{r_0}\right]^{n-1}(n+1), T = T_w, \quad at \quad z = 0,\\ &u = 0, v = v_0\left[\frac{r}{r_0}\right]^{2n-1}, T = T_\infty, \quad as \quad z \to \infty, \end{aligned} \tag{6}$$

The tangential flow v high above the planar surface at $z = 0$ is assumed to exhibit a power-law variation where the power $m = 2n - 1$ is a prescribed parameter. In the present study we also allow for suction through the surface. The suction velocity $w(r, 0)$ varies as r^{n-1} and $A < 0$ is a dimensionless suction parameter. The temperature varies from T_w at the surface to T_∞ in the vortex flow high above the surface.

By means of the usual boundary layer approximations, namely that $w \ll u,\ v$ and $\partial/\partial z \gg \partial/\partial r$, the system of Eqs. (1)–(5) simplifies to:

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{7}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial z^2}\right], \tag{8}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial z^2}\right], \tag{9}$$

$$\frac{\partial p}{\partial z} = 0. \tag{10}$$

$$\rho C_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) = k\frac{\partial^2 T}{\partial z^2}, \tag{11}$$

still subjected to the boundary conditions defined in Eq. (6). Here, Eq. (10) simply states that the pressure p remains constant across the three-dimensional boundary layer.

2.2. Generalized similarity transformation

In view of the generalized boundary conditions in Eq. (6), we now proceed and define the similarity transformations:

$$\begin{aligned} u(r,z) &= -v_0\left[\frac{r}{r_0}\right]^{2n-1}F(\eta),\\ v(r,z) &= v_0\left[\frac{r}{r_0}\right]^{2n-1}G(\eta),\\ w(r,z) &= v_0\sqrt{\frac{\nu}{v_0 r_0}}\left[\frac{r}{r_0}\right]^{n-1}[(n+1)H(\eta) + (n-1)\eta F],\\ p(r) &= \frac{\rho v_0^2}{4n-2}\left[\frac{r}{r_0}\right]^{4n-2},\\ T(r,z) &= T_\infty + (T_w - T_\infty)\Theta(\eta), \end{aligned} \tag{12}$$

where η is a dimensionless similarity variable defined by

$$\eta = \left[\frac{z}{r_0}\right]\left[\frac{r}{r_0}\right]^{n-1}\sqrt{\frac{v_0 r_0}{\nu}}. \tag{13}$$

In terms of the non-dimensional variables the governing equations become:

$$H' - F = 0, \tag{14}$$

$$F'' - (n+1)HF' - (1-2n)F^2 - G^2 + 1 = 0, \tag{15}$$

$$G'' - (n+1)HG' + 2nFG = 0, \tag{16}$$

$$\Theta'' - Pr(n+1)H(\eta)\Theta' = 0, \tag{17}$$

where Pr is the Prandtl number, $Pr = \rho\nu C_p/k$. The corresponding boundary conditions specified in (6) transform to:

$$\begin{aligned} &F(\eta) = 0, \quad G(\eta) = 0, \quad H(\eta) = A, \quad \Theta(\eta) = 1 \quad at \quad \eta = 0,\\ &F(\eta) = 0, \quad G(\eta) = 1, \quad \Theta(\eta) = 0 \quad as \quad \eta \to \infty. \end{aligned} \tag{18}$$

By means of the transformation defined in Eq. (12), the PDEs in Eqs. (7)–(11) transform exactly into a set of coupled non-linear ordinary differential equations (ODEs) subjected to the seven appropriate boundary conditions (18). This constitutes a three-parameter problem in terms of the power-law parameter $n = (m+1)/2$, the suction parameter A and the Prandtl number Pr.

2.3. Numerical approach

We solved the two-point boundary value problem consisting of the coupled set of ordinary differential Eqs. (14)–(17) subjected to the boundary conditions (18). For this purpose we have used the bvp4c MATLAB solver, which gives very good results for the non-linear ODEs with multipoint BVPs. This finite-difference code utilizes the 3-stage Lobatto IIIa formula, that is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth-order accurate uniformly in [a,b]. For multipoint BVPs, the solution is C1-continuous within each region, but continuity is not automatically imposed at the interfaces. Mesh selection and error control are based on the residual of the