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A novel model order reduction framework via staggered reduced basis space-time finite elements in linear first order transient systems

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ABSTRACT

A novel model order reduction framework for space and time domain discretizations is proposed. Iterative convergence of a Galerkin approximation in space and a Least Squares Petrov Galerkin approximation in time is obtained through a staggered reduced basis method in space-time. In every iteration, one of the two domains (space or time) is refined; and the other is reduced and a posteriori error indicators in space and time are used to drive the convergence iterations. Numerical results for 2D heat transfer and convection-diffusion problems demonstrate the significant computational efficiency of the proposed methodology. Comparisons of wall-clock times and solution accuracy with traditional time integration algorithms has been presented to validate the efficacy of the proposed framework and demonstrate computational savings of an order of magnitude.

1. Introduction

Partial differential equations, ubiquitious in a vast majority of fields in science are typically solved using numerical methods. The finite difference (FD) and the finite element (FE) methods are two of the most widely used techniques in computational mechanics. The finite element method, since its introduction, has become increasingly more popular and is largely preferred over the finite difference techniques in spatial discretizations. However, through a plethora of time integration algorithms, the FD type time integration is still the preferred technique in the temporal domain. Zienkiewicz et al. [1] introduced the weighted residual methodology in time to derive the SSpj (Single-step p,j method) and GNpj (Generalized Newmark p,j method) frameworks which incorporate several well known algorithms. Tamma et al. [2] further developed a unified generalized weighted residual methodology under the Generalized Single Step Singe Solve (GS4) I and II family of algorithms for first and second order equations in time respectively. Most past efforts over the past 50 years or so as related to LMS methods (to include the SSpj and GNpj frameworks and many more) and a much wider variety of new and optimal algorithms and designs as well have been designed under the umbrella of the Generalized Single Step Singe Solve (GS4) I and II framework for first and second order systems of equations in time respectively. The GS4 family of algorithms encompasses new advances with optimal algorithms encompassing the class of LMS methods that are all second-order time accurate including most well known time stepping schemes as subsets; and therein additionally combines first and second order time integration schemes through the use of a novel i-Integration framework applicable to a wide range of transient/dynamics systems of first or second order in time [3–7]. The equivalence between the single step finite difference, the linear multi-step (LMS) counterparts and finite element approximations in time was demonstrated in [2]. The preference to the FD representation stems largely from the ease of time marching and due to the extensive costs entailed by continuous finite element techniques in coupled space-time. Argyris and Scharpf [8] first introduced the application of the finite element in time for a second order dynamic system using Hermite cubic elements in time. The formulation was presented for single degree of freedom (SDOF) and further extended to the multi-degree (MDOF), through a kronecker product representation of space and time. This approach entails solving for the field variable at all DOFs and all time-points in a single solve. This so called continuous space-time finite element method, however can incur huge computational cost as the spatial (mesh) and/or the temporal (Δt) resolution is refined. Reed et al. [9] and Lesaint et al. [10] introduced a discontinuous Galerkin (DG) approach in time across constant temporal manifolds which was further developed by Hughes et al. [11] with an application to elastodynamics. The work by Richter [12] that utilizes fully unstructured DG approach in space-time and application of DG in elastodynamics by Yin et al. [13] are noteworthy.

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Model order reduction (MOR) techniques for low rank approximations in the spatial regime have been excessively researched and successfully developed over the last couple of decades. The proper orthogonal decomposition (POD) and the Greedy approach in model reduction of parametric PDEs, form the key ingredients of the certified reduced basis methods [14]. These techniques are however, largely utilized to approximate spatial manifolds and the process of identifying these manifolds for dynamic problems entails the high fidelity transient response of each parameter in the training parameter space. In addition, once the said subspace in space is identified, the reduced order model (ROM) is again subject to traditional time marching schemes. The reduction of the temporal basis has been largely unexplored. Urban and Patera [15,16] developed the application of certified reduced basis methods for space-time discretizations. The work in [15,16] focused on identifying approximating subspaces in space-time through high-fidelity solutions for a training parameter space. The work although unprecedented, requires a large number of high fidelity space-time solves which could prove computationally expensive.

For single parameter systems, traditional POD techniques for transient problems are capable of identifying a good approximation for a spatial basis by time-marching over a training time interval. The accuracy considerations of the POD procedure have been very well documented in [17]. However, no such methods have been identified in the literature to obtain a lower rank temporal approximation.

The work presented in this study proposes an iterative (in the sense of alternating or staggered) space-time convergence framework to obtain a lower dimensional approximation in both space and time. Independent discretizations in space and time are assumed which lead to structured space-time meshes. A Galerkin approach is considered in the full-order space discretization, whereas the projection in time follows a Least Squares Petrov Galerkin approach. The drawbacks of computational cost associated with continuous time discretizations are greatly alleviated through the proposed MOR Such a MOR procedure in time is not readily available with traditional time marching algorithms. In addition, as discussed in Section 2.2 the least squares projection in time inherits a time-continuous optimality in the L^2 sense in contrast to finite difference time marching schemes. A posteriori error estimates in space and time are developed to iteratively improve the low dimensional approximations.

2. Finite element framework in time

We present below, the Galerkin and Least squares projections from continuous to discrete time domains, however, depending on the test functions and representations in an LMS framework, many possible projections in time exist [18,19].

2.1. Galerkin formulation

Consider a SDOF initial value problem: find $u \in \mathbf{V_L}(I_t \equiv (0,T))$ such that,

$$Lu = f \tag{1}$$

$$u(0) = u_0 \tag{2}$$

where L is a linear differential operator in time. We introduce the variational form and seek the solution, $u \in \mathbf{V}(I_t)$, such that

$$A(\bar{u}, v) = (f, v) \quad \forall v \in \mathbf{V}(I_t)\; f \in \mathbf{V}\prime(I_t) \tag{3}$$

$$\bar{u} = u_0 + u \tag{4}$$

$$u(0) = 0$$

$$v(0) = 0$$

where, A is a bilinear form and f is a linear functional on V. The test and trial spaces coincide in the Galerkin approximation and are required to vanish on the boundary $t = 0$. The optimality of the Galerkin projection stems from the minimization of the energy norm. However, since the bilinear form A in time, is not self-adjoint in most ODEs, the optimality of the Galerkin projection, which is widely exploited for FEM in space, is lost in time [20]. We therefore prefer to work with the Least Squares projection in time due to associated optimality.

2.2. Least Squares Petrov Galerkin (LSPG) formulation

The LSPG approach inherits its optimality from the minimization of the L^2 norm of the residual which, for Eq. (1) is defined as,

$$R = Lu - f \tag{5}$$

$$u = \underset{u \in V_L}{\operatorname{argmin}} \|R\|_{L^2} = \underset{u \in V_L}{\operatorname{argmin}} \int_{I_t} (Lu - f)(Lu - f)dt \tag{6}$$

$$\therefore \delta\|R\|_{L^2} = \int_{I_t} \frac{\partial R}{\partial u} R dt = 0 \tag{7}$$

$$\therefore A(\bar{u}, v) = (f, v) \quad \forall v \in \mathbf{W}(I_t) \equiv \frac{\partial R}{\partial u} \tag{8}$$

The resulting set of relations on discretizing the above equation are symmetric and optimal in an L^2 sense.

2.3. High fidelity space-time discretizations

We develop the high fidelity space-time formulation based on a Galerkin projection in space and an LSPG projection in time by presenting its application to a model heat equation given in Eq. (9). The formulation in the presence of convective terms is easily extended.

$$\rho c \frac{\partial u}{\partial t} - \mu \Delta u = f \tag{9}$$

$$u = u_d(t) \quad \text{on} \Gamma_u \subset \Gamma \tag{10}$$

$$\frac{\partial u}{\partial n} = f_q(t) \quad \text{on} \Gamma \setminus \Gamma_u \tag{11}$$

where, Δ is the Laplace operator and ρ, c and μ are material parameters. The optimality of Galerkin projection in space and LSPG in time are independently preserved due to the structured space-time mesh. We discretize in space as follows,

$$u = u_d(t) + \phi_h u_h(t) \tag{12}$$

where we require $\phi_h = 0$ on Γ_u and $u_d(t) = \phi_{dh} u_{dh}(t)$ satisfies the Dirichilet boundary conditions. We formulate the time continuous spatial residual and perform a Galerkin projection as follows,

$$R(t) = \rho c(\phi_{dh}\dot{u}_{dh}(t) + \phi_h \dot{u}_h(t)) - \mu(\Delta\phi_{dh} u_{dh}(t) + \Delta\phi_h u_h(t)) - f(t) \tag{13}$$

$$\int_\Omega \phi_h^T R(t)\, d\Omega = 0 \tag{14}$$

$$\therefore \rho c \int_\Omega \phi_h^T \phi_{dh} \dot{u}_{dh}(t) d\Omega + \rho c \int_\Omega \phi_h^T \phi_h \dot{u}_h(t) d\Omega + \mu \int_\Omega \nabla\phi_h^T \nabla\phi_{dh} u_{dh}(t) d\Omega + \mu \int_\Omega \nabla\phi_h^T \nabla\phi_h u_h(t) d\Omega = \int_\Omega \phi_h^T f(x,t) d\Omega + \mu \int_{\Gamma\setminus\Gamma_u} \phi_h^T f_q(t) d\Gamma \tag{15}$$