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A novel numerical method for solving heat conduction problems

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ABSTRACT

In this work, an efficient numerical method with a high accuracy is proposed for solving the heat conduction problems. In this method, the governing equation of heat conduction in the partial differential equation form is firstly integrated over the small volume around each node point. In the resulting integrals the spatial derivatives of the unknown temperature and heat flux disappear. Then the numerical quadrature is employed to discretize the integrals. Numerical results show that when the same amount of the computer memory and CPU-time is consumed the proposed method can achieve a high accuracy in comparison with the finite volume method (FVM). Furthermore, the proposed method is more accurate than the finite element method (FEM) and boundary element method (BEM) for multi-dimensional heat conduction problems.

1. Introduction

The subject of heat conduction is of fundamental importance in many engineering applications, such as the thermal cooling, thermal protection, and heat exchange problems. The analytic solutions of heat conduction problems are limited by the complex geometries of the heat conduction media. As the rapid development of the computer technology, the numerical solution becomes a powerful approach for solving the heat conduction problems. Various numerical methods, such as the finite difference method (FDM) [1–4], FVM [5–10], FEM [11,12], BEM [13–16], and meshless method [17–19], have found wide applications in the heat conduction problems. Among them, the FVM is central to the most well-established CFD codes, such as CFX/ANSYS, FLUENT, PHOENICS and STAR-CD. The control volume integration which is implemented in the FVM can keep the conservation of the relevant properties for each finite size cell, which is one of main attractions of the FVM. Actually, the control volume integration has another advantage that it reduces the order of the highest derivative that appears in the governing equations of fluid flow and heat transfer, which weakens the requirement of the smoothness of the flow velocity and temperature field. Along this line, some integration methods based on the governing equations in the pure integral equation form have been developed. An axial Green's function method (AGM) was proposed for solving multi-dimensional elliptic boundary value problems [20]. Then it was applied to solve the Stokes flow [21]. Recently, a local axial Green's function method which is the localization of the AGM was developed for solving the convection–diffusion problems [22]. Similarly, a nonstandard finite difference scheme based on the Green's function formulation was established for solving the reaction–diffusion–convection problems [23]. Based on the Green's function in a series form and the integration formulation, we have proposed an integral equation approach for simulating the magnetic reconnection phenomenon [24] and the convection–diffusion problems [25]. The magnetic reconnection phenomenon involves the velocity, magnetic and temperature fields of the conducting plasmas. When the magnetic reconnection phenomena occur, the plasmas flows are usually turbulent. Therefore, numerical simulations of the three-dimensional magnetic reconnection phenomena require huge computer memory and CPU-time. It is of great value to develop efficient and robust numerical methods with high accuracy to simulate the magnetic reconnection phenomena. Although the magnetic reconnection is quite complicated, it mainly involves two processes, one is the diffusion (or heat conduction in the framework of heat transfer); the other is the convection. Thus a good numerical method for the magnetic reconnection must firstly solve the heat conduction problems efficiently and accurately. In the following, a novel numerical method for solving heat conduction problems is presented. The methodology of this method may be extended to develop efficient and robust numerical methods for simulations of the magnetic reconnection phenomena.

2. Governing equation and discretization

Consider the following steady heat conduction equation:

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$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)=S \tag{1}$$

where T is the temperature field, $(x,y,z)^T$ is the position coordinate in the Cartesian coordinate system, k_x, k_y and k_z are respectively the heat conductivities along x-, y- and z-directions, S is the heat source term. Eq. (1) can be rewritten into the following form:

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}=S \tag{2}$$

where q_x, q_y and q_z are the heat fluxes along x-, y- and z-direction, respectively. The Fourier law reads

$$q_x=-k_x\frac{\partial T}{\partial x} \tag{3a}$$

$$q_y=-k_y\frac{\partial T}{\partial y} \tag{3b}$$

$$q_z=-k_z\frac{\partial T}{\partial z} \tag{3c}$$

Denote the heat conduction medium as V. The node points $(x_i,y_j,z_k)^T$ $(i=0,\ 1,\ 2,\dots,M; j=0,\ 1,\ 2,\dots,N; k=0,\ 1,\ 2,\dots,K)$ are employed to divide the region V. Around the node point $(x_{i+1},y_{j+1},z_{k+1})^T$, select the small volume $\left[x_{i+\frac{1}{2}},x_{i+\frac{3}{2}}\right]\times\left[y_{j+\frac{1}{2}},y_{j+\frac{3}{2}}\right]\times\left[z_{k+\frac{1}{2}},z_{k+\frac{3}{2}}\right]$ here $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ and $z_{k+\frac{1}{2}}$ represent $x_i+0.5(x_{i+1}-x_i)$, $y_j+0.5(y_{j+1}-y_j)$, and $z_k+0.5(z_{k+1}-y_k)$, respectively. Note that the node point is located in the center of the small volume. Integrating Eq. (2) over the small volume yields

$$\begin{aligned}&-\int_{y_{j+\frac{1}{2}}}^{y_{j+\frac{3}{2}}}\int_{z_{k+\frac{1}{2}}}^{z_{k+\frac{3}{2}}}q_x\left(x_{i+\frac{3}{2}},y,z\right)-q_x\left(x_{i+\frac{1}{2}},y,z\right)dydz\\&-\int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}}\int_{z_{k+\frac{1}{2}}}^{z_{k+\frac{3}{2}}}q_y\left(x,y_{j+\frac{3}{2}},z\right)-q_y\left(x,y_{j+\frac{1}{2}},z\right)dxdz\\&-\int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}}\int_{y_{j+\frac{1}{2}}}^{y_{j+\frac{3}{2}}}q_z\left(x,y,z_{k+\frac{3}{2}}\right)-q_z\left(x,y,z_{k+\frac{1}{2}}\right)dxdy=S_{i+1,j+1,k+1}\end{aligned} \tag{4}$$

where $S_{i+1,j+1,k+1}=\int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}}\int_{y_{j+\frac{1}{2}}}^{y_{j+\frac{3}{2}}}\int_{z_{k+\frac{1}{2}}}^{z_{k+\frac{3}{2}}}Sdxdydz$. It is evident that after the integration there are no spatial derivatives of heat fluxes in the resulting Eq. (4). The integrals in Eq. (4) can be approximated by applying the mid-point formula, that is

$$\begin{aligned}&-\left[q_x\left(x_{i+\frac{3}{2}},y_{j+1},z_{k+1}\right)-q_x\left(x_{i+\frac{1}{2}},y_{j+1},z_{k+1}\right)\right]\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\left(z_{j+\frac{3}{2}}-z_{j+\frac{1}{2}}\right)\\&-\left[q_y\left(x_i,y_{j+\frac{3}{2}},z_{k+1}\right)-q_y\left(x_i,y_{j+\frac{1}{2}},z_{k+1}\right)\right]\left(x_{j+\frac{3}{2}}-x_{j+\frac{1}{2}}\right)\left(z_{j+\frac{3}{2}}-z_{j+\frac{1}{2}}\right)\\&-\left[q_z\left(x_{i+1},y_{j+1},z_{k+\frac{3}{2}}\right)-q_z\left(x_{i+1},y_{j+1},z_{k+\frac{1}{2}}\right)\right]\left(x_{j+\frac{3}{2}}-x_{j+\frac{1}{2}}\right)\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\\&=S_{i+1,j+1,k+1}\end{aligned} \tag{5}$$

Next integrating Eq. (3a) over $[x_i,x_{i+1}]$, we obtain:

$$\int_{x_i}^{x_{i+1}}q_xdx=-\int_{x_i}^{x_{i+1}}k_x\frac{\partial T}{\partial x}dx=-k_xT|_{x_i}^{x_{i+1}}+\int_{x_i}^{x_{i+1}}k'_xTdx$$

that is

$$\int_{x_i}^{x_{i+1}}q_xdx=-[k_x(x_{i+1},y)T(x_{i+1},y)-k_x(x_i,y)T(x_i,y)]+\int_{x_i}^{x_{i+1}}k'_xTdx \tag{6}$$

where $k'_x=\frac{\partial k_x}{\partial x}$. Note that there is no spatial derivatives of the temperature field and heat flux in Eq. (6). Applying the mid-point formula on the integral on the left side Eq. (6), and the trapezoid formula on the last integral on the right side, yield

$$\begin{aligned}q_x\left(x_{i+\frac{1}{2}},y,z\right)=&\frac{-k_x(x_{i+1},y,z)T(x_{i+1},y,z)+k_x(x_i,y,z)T(x_i,y,z)}{x_{i+1}-x_i}\\&+\frac{1}{2}(k'_x(x_{i+1},y,z)T(x_{i+1},y,z)+k'_x(x_i,y,z)T(x_i,y,z))\end{aligned} \tag{7}$$

If integrating Eq. (3b) over $[x_{i+1},x_{i+2}]$, the similar derivation gives rise to:

$$\begin{aligned}q_x\left(x_{i+\frac{3}{2}},y,z\right)=&\frac{-k_x(x_{i+2},y,z)T(x_{i+2},y,z)+k_x(x_{i+1},y,z)T(x_{i+1},y,z)}{x_{i+2}-x_{i+1}}\\&+\frac{1}{2}(k'_x(x_{i+2},y,z)T(x_{i+2},y,z)+k'_x(x_{i+1},y,z)T(x_{i+1},y,z))\end{aligned} \tag{8}$$

Similarly, from Eqs. (3b) and (3c) we obtain:

$$\begin{aligned}q_y\left(x,y_{j+\frac{1}{2}},z\right)=&\frac{-k_y(x,y_{j+1},z)T(x,y_{j+1},z)+k_y(x,y_j,z)T(x,y_j,z)}{y_{j+1}-y_j}\\&+\frac{1}{2}(k'_y(x,y_{j+1},z)T(x,y_{j+1},z)+k'_y(x,y_j,z)T(x,y_j,z))\end{aligned} \tag{9}$$

$$\begin{aligned}q_y\left(x,y_{j+\frac{3}{2}},z\right)=&\frac{-k_y(x,y_{j+2},z)T(x,y_{j+2},z)+k_y(x,y_{j+1},z)T(x,y_{j+1},z)}{y_{j+2}-y_{j+1}}\\&+\frac{1}{2}(k'_y(x,y_{j+2},z)T(x,y_{j+2},z)+k'_y(x,y_{j+1},z)T(x,y_{j+1},z))\end{aligned} \tag{10}$$

$$\begin{aligned}q_z\left(x,y,z_{k+\frac{1}{2}}\right)=&\frac{-k_z(x,y,z_{k+1})T(x,y,z_{k+1})+k_z(x,y,z_k)T(x,y,z_k)}{z_{k+1}-z_k}\\&+\frac{1}{2}(k'_z(x,y,z_{k+1})T(x,y,z_{k+1})+k'_z(x,y,z_k)T(x,y,z_k))\end{aligned} \tag{11}$$

$$\begin{aligned}q_z\left(x,y,z_{k+\frac{3}{2}}\right)=&\frac{-k_z(x,y,z_{k+2})T(x,y,z_{k+2})+k_z(x,y,z_{k+1})T(x,y,z_{k+1})}{z_{k+2}-z_{k+1}}\\&+\frac{1}{2}(k'_z(x,y,z_{k+2})T(x,y,z_{k+2})+k'_z(x,y,z_{k+1})T(x,y,z_{k+1}))\end{aligned} \tag{12}$$

Substituting Eqs. (7)–(12) into Eq. (5) and rearranging the resulting terms, yields

$$\begin{aligned}&a_{i+2,j+1,k+1}T_{i+2,j+1,k+1}+b_{i+1,j+1,k+1}T_{i+1,j+1,k+1}+c_{i,j+1,k+1}T_{i,j+1,k+1}\\&\quad+d_{i+1,j+2,k+1}T_{i+1,j+2,k+1}+e_{i+1,j,k+1}T_{i+1,j,k+1}+f_{i+1,j+1,k+2}T_{i+1,j+1,k+2}\\&\quad+g_{i+1,j+1,k}T_{i+1,j+1,k}=-S_{i+1,j+1,k+1}\end{aligned} \tag{13}$$

where

$$\begin{aligned}a_{i+2,j+1,k+1}=&\left[-k_{x,i+2,j+1,k+1}/(x_{i+2}-x_{i+1})+\frac{1}{2}k'_{x,i+2,j+1,k+1}\right]\\&\times\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\left(z_{k+\frac{3}{2}}-z_{k+\frac{1}{2}}\right)\end{aligned}$$

$$\begin{aligned}b_{i+1,j+1,k+1}=&[k_{x,i+1,j+1,k+1}/(x_{i+2}-x_{i+1})+k_{x,i+1,j+1,k+1}/\\&(x_{i+1}-x_i)]\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\left(z_{k+\frac{3}{2}}-z_{k+\frac{1}{2}}\right)+[k_{y,i+1,j+1,k+1}/\\&(y_{j+2}-y_{j+1})+k_{y,i+1,j+1,k+1}/\\&(y_{j+1}-y_j)]\left(x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}\right)\left(z_{k+\frac{3}{2}}-z_{k+\frac{1}{2}}\right)+[k_{z,i+1,j+1,k+1}/\\&(z_{k+2}-z_{k+1})+k_{z,i+1,j+1,k+1}\\&(z_{k+1}-z_k)]\left(x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}\right)\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\end{aligned}$$

$$\begin{aligned}c_{i,j+1,k+1}=&[-k_{x,i,j+1,k+1}/(x_{i+1}-x_i)\\&-0.5k'_{x,i,j+1,k+1}]\left(y_{j+\frac{3}{2}}-y_{j+\frac{1}{2}}\right)\left(z_{k+\frac{3}{2}}-z_{k+\frac{1}{2}}\right)\end{aligned}$$