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Accurate empirical formulas for the evaluation of origin intensity factor in singular boundary method using time-dependent diffusion fundamental solution

Fajie Wang, Wen Chen *

State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing 210098, China

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ABSTRACT

This paper presents the simple empirical formulas for the accurate evaluation of the origin intensity factor in singular boundary method (SBM) when the time-dependent diffusion fundamental solution is employed. These new formulas makes the SBM more fast, straightforward, and efficient for transient diffusion problems while being truly meshless, integration-free and easy-to-implement. Three benchmark examples are tested to demonstrate the accuracy and efficiency of the proposed scheme. It is shown that the SBM using these empirical formulas works well especially for one and two-dimensional transient diffusion problems. In the three-dimensional case, we have obtained the SBM empirical formula of the origin intensity factors at initial time, and numerical experiments on benchmark problems have verified its efficiency and accuracy. It is worth noting that we need to two different formulas for a specific dimensionality. However, the empirical formula with time variation in three-dimensional case is not available and still under investigation.

1. Introduction

Diffusion equation models a huge variety of physical problems in many areas of science and engineering applications and has long been a very attractive area of research. The boundary element method (BEM) [1–6] is a powerful and efficient computational method for the solution of such problems, thanks to its merits of boundary-only discretization and semi-analytical nature. Most of the BEM schemes employ time-independent, i.e., steady-state fundamental solution, to diffusion equations as the weight functions, but require treating the time derivative term by using the Laplace transform [2,3,6] or the finite difference scheme [1], which can be time-consuming and computationally expensive. Such schemes involve domain integrals [7] and spoil the boundary discretization merit. Consequently, the attractiveness of the BEM is largely lost with the time-independent fundamental solution for transient diffusion equation.

The singular boundary method (SBM) [8–10] is a relatively new meshless boundary collocation method for the numerical solution of boundary/initial value problems governed by certain partial differential equations. The method involves a coupling between the indirect BEM [11–14] and the method of fundamental solutions (MFS) [15,16]. The main idea is to fully inherit the dimensionality and stability superiorities of the BEM and the meshless and integration-free properties of the MFS. The advantages of the SBM over the more classical domain or boundary discretization methods can be concluded as follows:

(a) The SBM, as a boundary-type method, shares the same advantages of the BEM over domain discretization methods.
(b) The SBM does not require the task of domain and/or boundary meshing which may otherwise be arduous, time-consuming and computationally expensive for problems in complex geometries and high dimensions.
(c) The SBM is not involved with costly integrations which may be troublesome in the case of the BEMs.
(d) The SBM sidesteps the perplexing fictitious boundary issue associated with the traditional MFS [16–18], while retaining mathematically simple, easy-to-program and truly meshless.

In recent years, the SBM has been successfully applied to the potential [19], elasticity [20], Helmholtz [10,21], poroelastic wave [22], Stokes flow [23], etc. Nevertheless, these studies only employ the steady-state fundamental solution rather than time-dependent

* Corresponding author. Fax: +86 25 83736860.
E-mail address: chenwen@hhu.edu.cn (W. Chen).

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fundamental solution. Very recently, the method has been extended to transient diffusion problems using the time-dependent fundamental solutions [24]. The origin intensity factor (OIF) that isolates the singularities of the fundamental solutions is evaluated by the so-called inverse interpolation technique in [24], which requires selecting sample points and solving equations. The task can be computationally expensive and numerically unstable.

Based on the observation and fitting of the diagonal elements of the SBM interpolation matrix obtained by using the inverse interpolation technique, this study proposes the empirical formulas to accurately evaluate the OIF using time-dependent fundamental solution. It is shown that the proposed formulas makes the SBM more fast, straightforward, and efficient since little effort is required to calculate the OIF.

The outline of the rest of this paper is as follows. Section 2 introduces the governing equations and the boundary/initial conditions of diffusion problems. And then the SBM formulation and the empirical formulas for the determination of the OIF are provided in Section 3. Numerical experiments and discussions are provided in Section 4. Finally, some conclusions and remarks are drawn in Section 5.

2. Mathematical formulation

Consider a linear diffusion equation in an open bounded domain Ω, and assume that Ω is bounded by a boundary $\partial\Omega = \Gamma$. The governing equation is given by

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t} = k\nabla^2 u(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Omega, \tag{1}$$

where $\boldsymbol{x}$ is the general spatial coordinate, t the time, k the diffusion coefficient, and u the scalar variable to be determined. The initial condition of the diffusion problem is

$$u(\boldsymbol{x},t_0) = \bar{u}_0, \quad \boldsymbol{x} \in \Omega, \tag{2}$$

and the Dirichlet and Neumann boundary conditions are

$$u(\boldsymbol{x},t) = \bar{u}(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \Gamma_D, \tag{3}$$

$$q(\boldsymbol{x},t) = \frac{\partial u(\boldsymbol{x},t)}{\partial \boldsymbol{n}} = \bar{q}(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \Gamma_N, \tag{4}$$

where Γ_D and Γ_N indicate the parts of which the Dirichlet and Neumann boundary conditions are prescribed, respectively. t_0 represents initial time, $\boldsymbol{n}$ denotes the outward normal vector, the overline quantities $\bar{u}_0$, $\bar{u}$ and $\bar{q}$ indicate the given values.

The fundamental solution of governing equation (1) is given by

$$u^*(\boldsymbol{x},t;\xi,\tau) = \frac{e^{\frac{-|\boldsymbol{x}-\xi|^2}{4k(t-\tau)}}}{[4\pi k(t-\tau)]^{n/2}} H(t-\tau), \tag{5}$$

where $|\boldsymbol{x}-\xi|$ denotes the Euclidean distance between points $\boldsymbol{x}$ and ξ in R^n, n is the spatial dimension number, and $H(t)$ the Heaviside step function.

3. The singular boundary method using time-dependent fundamental solution of diffusion equation

3.1. The SBM formulation

Similar to the MFS, the SBM uses the time-dependent fundamental solution of diffusion equation as the basis function of its interpolation. Unlike the MFS, the SBM sidesteps the perplexing fictitious boundary issue associated with the former by means of the introduction of origin intensity factors to isolate the singularities of the fundamental solutions and to allow the coincidence of the source and collocation points as shown in Fig. 1. With this idea in mind, the solution can be approximated by a linear combination of fundamental solution as follows

$$u(\boldsymbol{x}_i,t_i) = \sum_{j=1,\, j\neq i}^{N=N_1+N_2} \alpha_j u^*(\boldsymbol{x}_i,t_i;\xi_j,\tau_j) + \alpha_i u_{ii}, \tag{6}$$

where $\boldsymbol{x}_i$ represents the location of the field points, and ξ_j gives the location of the source points, t_i and τ_j are the time of the field and source points, respectively, N_1 and N_2 the number of initial and boundary source points, and α_j the undetermined coefficients. u_{ii} are defined as the origin intensity factors, i.e., the diagonal elements of the SBM interpolation matrix. It is observed that for $\boldsymbol{x}_i = \xi_j$ and $t_i = \tau_j$, u_{ii} are singular, a fact that requires some special treatments. The key point in achieving the required accuracy and efficiency of the SBM is the accurate evaluation of the origin intensity factors.

3.2. Inverse interpolation technique for origin intensity factors

The origin intensity factors u_{ii} for Dirichlet boundary conditions present a weak singularity and can be calculated by using the inverse interpolation technique as summarized below.

Step 1. Let us assume a pure Dirichlet problem with all the boundary values set as $\bar{u}_s(\boldsymbol{x},t)$, where $\bar{u}_s(\boldsymbol{x},t)$, named as sample solution hereafter in this paper, is an arbitrary known particular solution, such as

$$\bar{u}_s(x,t) = \sin(x)e^{-kt} + 1, \quad \text{for } 1D, \tag{7}$$

$$\bar{u}_s(x,y,t) = (\sin(x) + \sin(y))e^{-kt} + 1, \quad \text{for } 2D, \tag{8}$$

$$\bar{u}_s(x,y,z,t) = (\sin(x) + \sin(y) + \sin(z))e^{-kt} + 1, \quad \text{for } 3D. \tag{9}$$

And then some sample points $\boldsymbol{y}_k$ need to be placed inside the physical domain. It is noted that the sample points $\boldsymbol{y}_k$ do not coincide with the source points ξ_j, and the sample points number N_k should not be fewer than the physical boundary source node number N. In this study, we choose $N_k = N$ sample points which coincided with the source points in the spatial position, but at different time levels.

Step 2. Using the interpolation formula (6), we can then determine the influence coefficients β_j by the following linear equations

$$u_s(\boldsymbol{y}_k,t_k) = \sum_{j=1}^{N_k} \beta_j u^*(\boldsymbol{y}_k,t_k;\xi_j,\tau_j). \tag{10}$$

Step 3. Replacing the sample points $\boldsymbol{y}_k$ with the boundary collocation points $\boldsymbol{x}_i$, the SBM interpolation matrix of the diffusion problem can be written as

$$u_s(\boldsymbol{x}_i,t_i) = \sum_{j=1,\, j\neq i}^{N_k} \beta_j u^*(\boldsymbol{x}_i,t_i;\xi_j,\tau_j) + \beta_i u_{ii}. \tag{11}$$

It is noted that only the origin intensity factors u_{ii} are unknown in the above equation. Thus, the origin intensity factor u_{ii} can be calculated via

$$u_{ii} = \frac{1}{\beta_j}\left[u_s(\boldsymbol{x}_i,t_i) - \sum_{j=1,\, j\neq i}^{N_k} \beta_j u^*(\boldsymbol{x}_i,t_i;\xi_j,\tau_j)\right], \quad i = 1,2,\ldots,N. \tag{12}$$

Using the procedure described above, the origin intensity factors on the Dirichlet boundary conditions of the diffusion problems have been extracted out, as shown in Eq. (12).