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Identification of thermal conductivity coefficient and volumetric heat capacity of functionally graded materials

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ABSTRACT

Functionally graded materials are being widely implemented in many fields of technology. In the paper we present an approach to solving an inverse problem on a simultaneous identification of thermal conductivity coefficient and volumetric heat capacity of functionally graded materials. As an input data we use measurements of temperature and heat flux at body's boundary. The inverse problem solving is reduced to an iterative procedure when each iteration provided solving of the system of Fredholm's integral equations of the first kind. As an example, the problem for a rod is considered. The direct problem is solved by means of the Galerkin method. Examples of simultaneous reconstruction of different inhomogeneous laws of thermalphysic characteristics are presented.

1. Introduction

In recent years, composites with continuous change of physical properties called functionally graded materials (FGM) have been increasingly incorporated in different fields of technology. Unlike layered composites, FGM make it possible to avoid jumps in thermalphysical characteristics through an interfacial area [1]. To describe adequately thermal processes occurring in FGM, it is necessary to refuse a hypothesis of homogeneity. It is worth noting that a material can also become inhomogeneous during operation, for instance, with large temperature drops.

In case of inhomogeneous bodies, direct measurements of thermalphysical properties (thermal conductivity coefficient and volumetric heat capacity) are impossible as they present some coordinate functions. But if there is a thermal process in a medium, then on the basis of measurable input data it is possible to reconstruct a structure of the medium by means of solving coefficient inverse problem of thermal conductivity (CIPTC). In practice they often use two types of CIPTC statements. For the first type of statement, we assume that a data on a temperature is known in all the surface and inner points of the body for some time point [2]. For the second type, the data on temperature and heat flows is known just at some boundary part for some time interval [3].

As a rule, CIPTC are ill-posed and nonlinear problems, therefore, it is vital to build stable computational algorithms of their solving. The most widespread approach to solving CIPTC is based on its reduction to a minimization of residual functional. To minimize it, they often use gradient methods [2,4–12] or genetic algorithms [13].

In the number of papers the alternative approaches to solving CIPTC are proposed: quasiinversion method [14], statistical inversion approach [15], iterative process containing solving Fredholm's integral equations of the first kind.

When solving CIPTC, you are often limited by a restriction on a reconstruction: you can reconstruct only the thermal conductivity coefficient. The papers [2,5,6,8,16–21] are devoted to finding the thermal conductivity coefficient. However, in practice all of the thermalphysic characteristics are usually unknown. In the problem of their simultaneous identification, an important aspect is to state boundary conditions and additional data properly in order to provide the uniqueness of the inverse problem solution. In this way, in [22,23], the authors considered and mathematically grounded some types of statements of heat-conduction inverse problems providing uniqueness of the solution in the framework of a single experiment.

Unlike [22,23], in the present paper we deal with the procedure of simultaneous reconstruction of two thermalphysic characteristics

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of FGM in the frameworks of two experiments. To solve nonlinear inverse problem on the basis of the approach developed in [3,24–28], we obtained some operator equations allowing building an iterative process, at every step of which a linear problem had to be solved. One of the efficient ways of linearization is using weak statements of the direct problem [29–33]. As an example of an application of such an approach, we considered the inverse problem for a rod. The direct problem was solved by the Galerkin method. We conducted a series of computational experiment on the simultaneous identification of thermalphysic characteristics of the rod.

2. Operator equations for solving inverse coefficient problem of thermal conductivity

Let us consider the problem of heat distribution in a limited inhomogeneous body. The initial boundary value problem has form

$$(k_{ij}(M)T(M,t)_{,j})_{,i} = c(M)\dot{T}(M,t), \tag{1}$$

$$T|_{S_T} = 0, -k_{ij}T_{,i}n_j|_{S_q} = q, \tag{2}$$

$$T(M,0) = 0. \tag{3}$$

Here k_{ij} are the components of the thermal conductivity tensor, c is the volumetric heat capacity, T is the body temperature, n_j are the components of the unit vector of outer normal to S_q, q is the heat flux density.

The direct problem of thermal conductivity is to find the function T from (1)–(3) when the thermalphysic characteristics (k_{ij}, c) are known.

In the inverse problem it is required to define the thermalphysic characteristics (k_{ij}, c) by an additional data on the temperature measured at the body's boundary part:

$$T|_{S_T} = f(M,t), t \in [T_1, T_2]. \tag{4}$$

The stated inverse problem (1)–(4) is nonlinear. The weak statement of the direct problem (1)–(3) allows formulating an iterative process and operator equations at its each step without a complicated procedure of calculation of the Frechet derivatives of nonlinear operators [24].

Let us introduce the test function θ satisfying the boundary condition $\theta|_{S_T} = 0$. Then the weak statement of the problem can be written in the following form [34]:

$$\int_V [(k_{ij}T_{,j})_{,i} - A\dot{T}]\theta dV = 0, \tag{5}$$

By applying the Ostrogradski–Gauss theorem to (5) and using the boundary conditions (2), we have

$$\int_V k_{ij}T_{,i}\theta_{,j}dV + \int_V c\dot{T}\theta dV = \int_{S_q} q\theta dS. \tag{6}$$

The formula (6) can be written as:

$$(a, T, \theta) = B(\theta). \tag{7}$$

Here

$A(a,T,\theta) = \int_V k_{ij}T_{,i}\theta_{,j}dV + \int_V c\dot{T}\theta dV$ is the trilinear form, i.e. the form which is linear by the coefficients $a(k_{ij}, c)$ of the differential Eq. (1), the temperature T and the test function θ;

$B(\theta) = \int_{S_q} q\theta dS$ is the linear form for the test function θ.

Let us build an iterative process based on the relation (7). The iterative process starts from the initial approximation of the coefficients $a^{(0)}$. Denote $T^{(0)}$ as the corresponding field satisfying the weak statement $(a^{(0)}, T^{(0)}, \theta) = B(\theta)$. Then the sequence of problems to define $T^{(n-1)}$ in accordance with [24] takes form $(a^{(n-1)}, T^{(n-1)}, \theta) = B(\theta)$, and the correction $\delta a^{(n-1)} = a - a^{(n-1)}$ is to be found from the operator equation of the first kind with compact kernel:

$$(\delta a^{(n-1)}, T^{(n-1)}, \theta^{(n-1)}) = B(f - f^{(n-1)}). \tag{8}$$

In expanded form, the Eq. (8) takes form

$$\int_V \delta k_{ij}^{(n-1)}(T_{,i}^{(n-1)})^2 dV + \int_V \delta c^{(n-1)}\dot{T}^{(n-1)}T^{(n-1)}dV$$
$$= \int_{S_q} q(f - T^{(n-1)})dS, t \in [T_1, T_2]. \tag{9}$$

However, just one Eq. (9) is not enough to find corrections for two unknown functions. It is necessary to get the second equation proceeding from the thermal conduction problem with other boundary conditions.

As a specific example of an application of the proposed approach, let us consider the problem of simultaneous reconstruction of thermalphysic characteristics of an inhomogeneous rod.

3. Thermal conductivity problem for functionally graded rod

Let us consider a problem on heat distribution in a straight inhomogeneous isotropic rod of the length l with zero maintained temperature at one end $(x = 0)$ and constant heat flux q_0 at the other one $(x = l)$. The initial conditions are assumed to be zero:

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}\right) = c(x)\frac{\partial T}{\partial t}, 0 \leqslant x \leqslant l, t > 0, \tag{10}$$

$$T(0,t) = 0, -k(l)\frac{\partial T}{\partial x}(l,t) = q_0, \tag{11}$$

$$T(x,0) = 0. \tag{12}$$

Let us switch to the dimensionless parameters and functions in (10)–(12), denoting $z = \frac{x}{l}, \bar{k}(z) = \frac{k(x)}{k_0}, \bar{c}(z) = \frac{c(x)}{c_0}, \tau = \frac{k_0 t}{c_0 l^2}, W(z,\tau) = \frac{k_0 T}{q_0 l}, k_0 = \max_{x\in[0,l]} k(x), c_0 = \max_{x\in[0,l]} c(x)$.

After it, the initial boundary value problem (10)–(12) will take form:

$$\frac{\partial}{\partial z}\left(\bar{k}(z)\frac{\partial W}{\partial z}\right) = \bar{c}(z)\frac{\partial W}{\partial \tau}, 0 \leqslant z \leqslant 1, \tau > 0, \tag{13}$$

$$W(0,\tau) = 0, -\bar{k}(1)\frac{\partial W}{\partial z}|_{z=1} = 1, \tag{14}$$

$$W(z,0) = 0. \tag{15}$$

The direct problem of thermal conduction is to find function $W(z,\tau)$ from (13)–(15) when thermalphysical characteristics $\bar{c}(z)$ and $\bar{k}(z)$ are known.

In the inverse problem it is required to recover simultaneously both $\bar{c}(z)$ and $\bar{k}(z)$ by the additional data on temperature value at the rod's end $z = 1$ for some time interval:

$$W(1,\tau) = f(\tau), \tau \in [a,b]. \tag{16}$$

To solve the inverse problem, we use the operator Eq. (9) which in case of the rod transfers into the following dimensionless form:

$$\int_0^1 \delta\bar{k}^{(n-1)}M_1(z,\tau)dz + \int_0^1 \delta\bar{c}^{(n-1)}M_2(z,\tau)dz = F_1(\tau), \tau \in [a,b]. \tag{17}$$

Here, the kernels and the right part of the Eq. (17) have form

$$M_1(z,\tau) = \left(\frac{\partial W^{(n-1)}}{\partial z}\right)^2, \; M_2(z,\tau) = \frac{\partial W^{(n-1)}}{\partial \tau}W^{(n-1)},$$

$$F_1(\tau) = f(\tau) - W^{(n-1)}(1,\tau).$$