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The time differential three-phase-lag heat conduction model: Thermodynamic compatibility and continuous dependence

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ABSTRACT

This paper deals with the time differential three-phase-lag heat transfer model aiming, at first, to identify the restrictions that make it thermodynamically consistent. The model is thus reformulated by means of the fading memory theory, in which the heat flux vector depends on the history of the thermal displacement gradient: the thermodynamic principles are then applied to obtain suitable restrictions involving the delay times. Consistently with the thermodynamic restrictions just obtained, a first result about the continuous dependence of the solutions with respect to the given initial data and to the supply term is established for the related initial boundary value problems. Subsequently, to provide a more comprehensive review of the problem, a further continuous dependence estimate is proved, this time conveniently relaxing the hypotheses about the above-said thermodynamic restrictions. This last estimate allows the solutions to grow exponentially in time and so to have asymptotic instability.

1. Introduction

Over the last decades, much attention has been devoted to the theory originally proposed by Tzou [1–3] about the so-called dual-phase-lag heat conduction model. Such a theory essentially replaces the classical Fourier law with the following constitutive equation

$$q_i(\mathbf{x}, t+\tau_q) = -k_{ij}(\mathbf{x}) T_{,j}(\mathbf{x}, t+\tau_T), \quad \text{with} \quad \tau_q, \tau_T > 0 \tag{1}$$

stating, synthesizing its meaning, that the temperature gradient $T_{,j}$ at a certain time $t+\tau_T$ results in a heat flux vector q_i at a different time $t+\tau_q$. In the above constitutive equation (1), besides the explicit dependence upon the spatial variable, we point out that q_i are the components of the heat flux vector, T represents the temperature variation from the constant reference temperature $T_0 > 0$ and k_{ij} are the components of the conductivity tensor; moreover, t is the time variable while τ_q and τ_T are the phase lags (or delay times) of the heat flux and of the temperature gradient, respectively. In particular, τ_q is a relaxation time connected to the fast-transient effects of thermal inertia, while τ_T is caused by microstructural interactions, such as phonon scattering or phonon–electron interactions [4].

We emphasize that the related time differential models (obtained considering the Taylor series expansions of both sides of Eq. (1) and retaining terms up to suitable orders in τ_q and τ_T) have been widely investigated with respect to their thermodynamic consistency as well as to interesting stability issues (see, for example, [5–7]).

A natural evolution of the dual-phase-lag heat conduction model by Tzou consisted in the addition, by Roy Choudhuri [8], of a third delay time τ_α, which has led to a three-phase-lag heat conduction theory. He took into account the model by Green and Naghdi [9–12] which includes, among the constitutive variables, not only the temperature gradient but also the thermal displacement gradient. Starting from the Green–Naghdi model, Roy Choudhuri [8] proposed the following constitutive equation for the heat flux vector

$$q_i(\mathbf{x}, t+\tau_q) = -k_{ij}(\mathbf{x}) T_{,j}(\mathbf{x}, t+\tau_T) - K_{ij}(\mathbf{x}) \alpha_{,j}(\mathbf{x}, t+\tau_\alpha), \tag{2}$$

where α is the thermal displacement variable, being T equal to the partial time derivative of α, K_{ij} is a thermal tensor characteristic of the considered theory and τ_α is a new phase lag related to the thermal displacement gradient $\alpha_{,i}$: we can suppose, for example, that $0 \leqslant \tau_\alpha < \tau_T < \tau_q$. Through this equation, that generalizes Eq. (1), once again a lagging behavior is described. In agreement with the

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Tzou's interpretation, Eq. (2) means that a temperature gradient and a thermal displacement gradient imposed across a volume element at times $t+\tau_T$ and $t+\tau_\alpha$, respectively, result in a heat flux flowing at a different time $t+\tau_q$.

Also in this case, exactly as for the constitutive equation by Tzou, time differential (three-phase-lag) models can be considered, obtained through the Taylor series expansions of both sides of Eq. (2) and retaining terms up to suitable orders in τ_q, τ_T and τ_α. At this regard, in Quintanilla [13] and Quintanilla and Racke [14,15] it is possible to find some interesting references to the Taylor expansion orders issue.

In the present work and with regard to Eq. (2), the terms up to the second order in τ_q and up to the first order in τ_T and τ_α are retained, leading to the following generalized heat conduction law valid at the position $\mathbf{x}$ and at the time instant t:

$$\frac{1}{2}\tau_q^2\ddot{q}_i(\mathbf{x},t)+\tau_q\dot{q}_i(\mathbf{x},t)+q_i(\mathbf{x},t)=-\tau_T k_{ij}(\mathbf{x})\dot{T}_{,j}(\mathbf{x},t) - \left[k_{ij}(\mathbf{x})+\tau_\alpha K_{ij}(\mathbf{x})\right]T_{,j}(\mathbf{x},t)-K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x},t). \quad (3)$$

The purpose of this paper is twofold: on one side, following [6], we want to reformulate the constitutive equation (3) in such a way that the heat flux vector q_i depends on the history of the thermal displacement gradient, in order to evaluate the thermodynamic consistency of the considered time differential three-phase-lag model. To this end, we rewrite Eq. (3) in the framework of Gurtin–Pipkin [16] and Coleman–Gurtin [17] fading memory theory, and on this basis we analyze the compatibility of the model with the thermodynamical principles. Subsequently, precisely under the thermodynamic compatibility hypotheses just found, we prove the continuous dependence of the solutions from the initial data and from the external heat supply. A uniqueness theorem is also readily obtained as a direct consequence of these results. Finally, to provide a more complete overview about the issue in question, a further continuous dependence estimate is established under a suitable assumption which relaxes the previous thermodynamic compatibility hypotheses. This last estimate allows the solutions to grow exponentially in time and so one can be led to an unstable system.

2. The basic mathematical model

In this paper, referring to a fixed system of rectangular Cartesian axes Ox_k, $(k=1,2,3)$, we employ the usual summation and differentiation conventions. For the components of tensors of order $p\geqslant 1$, the Latin subscripts range over the set $\{1,2,3\}$, while a superposed dot or a subscript preceded by a comma denote partial differentiation with respect to the time variable t or to the corresponding Cartesian coordinate x_i, respectively; summation is understood over the repeated subscripts. Moreover, with an overlying bar we want to denote the closure of the corresponding set indicated below it.We suppose to deal with a regular region B, whose boundary is denoted by ∂B, and consider the linear theory of the time differential three-phase-lag heat conduction model as formulated through the following set of equations: the energy equation

$$-q_{i,i}+\rho r=c\ddot{\alpha}, \quad \text{in} \quad B\times(0,\infty), \quad (4)$$

the constitutive equation

$$q_i+\tau_q\dot{q}_i+\frac{1}{2}\tau_q^2\ddot{q}_i=-\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\beta}_j-\tau_T k_{ij}\ddot{\beta}_j-K_{ij}\beta_j, \ \text{in} \ \overline{B}\times[0,\infty), \quad (5)$$

the geometrical equation

$$\beta_j=\alpha_{,j}, \quad \text{in} \quad \overline{B}\times[0,\infty). \quad (6)$$

For a greater clarity, let us repeat some concepts already shown in the above Introduction, representing here all the notations used: q_i are the components of the heat flux vector, ρ is the mass density of the considered medium, r is the external heat supply per unit mass, c is the specific heat and α is the thermal displacement, being $T=\dot{\alpha}$ the temperature variation from the constant reference temperature $T_0>0$. The components of the thermal displacement gradient vector are denoted by β_i and we also recall that k_{ij} are the components of the conductivity tensor and K_{ij} are the components of a thermal tensor characteristic of the considered theory.

Further, we define the initial boundary value problem $\mathcal{P}$ by the basic equations (4)–(6), the following initial conditions

$$\begin{aligned}&\alpha(\mathbf{x},0)=0, \quad \dot{\alpha}(\mathbf{x},0)=T^0(\mathbf{x}),\\ &q_i(\mathbf{x},0)=q_i^0(\mathbf{x}), \quad \dot{q}_i(\mathbf{x},0)=\dot{q}_i^0(\mathbf{x}), \quad \text{in } \overline{B},\end{aligned} \quad (7)$$

recalling that $\alpha(\mathbf{x},0)=0$ because

$$\alpha(\mathbf{x},t)=\int_0^t T(\mathbf{x},s)ds,$$

as well as the following boundary conditions

$$\alpha(\mathbf{x},t)=\omega(\mathbf{x},t), \quad \text{on} \quad \overline{\Sigma}_1\times[0,\infty),$$

$$q_i(\mathbf{x},t)n_i=q(\mathbf{x},t), \quad \text{on} \quad \Sigma_2\times[0,\infty),$$

where $\overline{\Sigma}_1\cup\Sigma_2=\partial B$ and $\Sigma_1\cap\Sigma_2=\emptyset$ and having denoted by q_in_i the heat flux at any regular point of ∂B. We assume that the initial data $T^0(\mathbf{x}), q_i^0(\mathbf{x}), \dot{q}_i^0(\mathbf{x})$ and the boundary data $\omega(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are continuous prescribed functions selected in such a way to guarantee the existence of reciprocal compatibility conditions in $t=0$ and on ∂B.

Let us call $S=\{\alpha,q_i\}$, with $\alpha(\mathbf{x},t)\in C^{1,2}(B\times(0,\infty))$ and $q_i(\mathbf{x},t)\in C^{1,2}(B\times(0,\infty))$, a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $D=\left\{r;T^0,q_i^0,\dot{q}_i^0;\omega,q\right\}$.

3. Thermodynamic consistency of the model

Following the example of Fabrizio and Lazzari [6], we want to rewrite Eq. (5) as a memory constitutive equation of the type described in Gurtin–Pipkin [16] and Coleman–Gurtin [17]. In order to do this, let us rewrite it in terms of the thermal displacement variable α $(T=\dot{\alpha})$:

$$\frac{1}{2}\tau_q^2\ddot{q}_i(t)+\tau_q\dot{q}_i(t)+q_i(t)=-\tau_T k_{ij}\ddot{\alpha}_{,j}(t)-\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)-K_{ij}\alpha_{,j}(t) \quad (8)$$

and then solve it as a linear non-homogeneous second-order differential (in time) equation. We immediately see that the homogeneous (complementary) solution has the form

$$q_i^0(t)=C_i^c\left(\exp\frac{-t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)+C_i^s\left(\exp\frac{-t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right). \quad (9)$$

Through the application of the method of variation of constants, we aim to find a couple of functions $K_i^c(t)$ and $K_i^s(t)$ so that

$$q_i^*(t)=K_i^c(t)\left(\exp\frac{-t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)+K_i^s(t)\left(\exp\frac{-t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right) \quad (10)$$

is a solution of Eq. (8). After appropriate differentiations and straightforward calculations, the problem is reduced to the study of a system in the variables $\dot{K}_i^c(t)$ and $\dot{K}_i^s(t)$, providing

$$\begin{cases}\dot{K}_i^c(t)=\frac{2}{\tau_q}\left(\exp\frac{t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(t)+\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)+K_{ij}\alpha_{,j}(t)\right]\\ \dot{K}_i^s(t)=-\frac{2}{\tau_q}\left(\exp\frac{t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(t)+\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)+K_{ij}\alpha_{,j}(t)\right]\end{cases}$$