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High-resolution high-order upwind compact scheme-based numerical computation of natural convection flows in a square cavity

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ABSTRACT

In this paper, a higher-order accuracy method is proposed for the solution of time-dependent nature convection problems based on the stream function-vorticity form of Navier–Stokes equations, in which an optimized third-order upwind compact scheme (Opt-UCD3) with high resolution is proposed to approximate the nonlinear convective terms, the fourth-order symmetrical Padé compact scheme is utilized to discretize the viscous terms, the fourth-order compact scheme on the nine-point 2D stencil is used for approximating the stream-function Poisson-type equation and the third-order TVD Runge–Kutta method is employed for the time discretization. To assess numerical capability of the newly proposed algorithm, particularly its spatial behavior, a problem with analytical solution and another one with a steep gradient are numerically solved. Moreover, the nature convection flows in the square cavity with adiabatic horizontal walls and differentially heated vertical walls are also computed for the wide range of Rayleigh numbers ($10^3 < Ra < 10^{10}$). The characteristic parameters such as Nusselt number, velocity, and streamline show excellent agreement with benchmark solutions and some accurate results available in the literature. Additionally, the detailed features of flow phenomena for the higher Rayleigh numbers ($10^8 < Ra < 10^{10}$) are delineated. The results show that the natural convection flow looses stability firstly via a Hopf bifurcation at Ra_{c1} to the periodic flow regime, and then undergoes second bifurcation at a critical Rayleigh number Ra_{c2} to quasi-periodic flow regime, and eventually transits to turbulent through a further bifurcation. In the periodic regime, there exist at least two branches of solutions. All of the results are agree well with ones in the literature and show the capabilities of the present method to properly simulate the unsteady nature convection problems.

1. Introduction

In the past few decades, natural convection flow in a square cavity with differentially heated vertical walls and adiabatic horizontal walls has been studied extensively due to its fundamental importance in the understanding of buoyancy-driven flows and its relevance to a wide range of engineering applications, such as in cooling of nuclear reactors, air conditioning of rooms, cooling of electronic equipment, crystal growing of liquids, solar energy collector, etc. It has been a commonly studied problem of heat transfer and fluid mechanics because of its numerical and experimental accessibility and its simple geometry, and also has become one of the most popular case for verifying new computer programs and testing numerical algorithms developed to solve the Navier–Stokes (NS) equations.

Following the pioneering work by Batchelor [1] and the popularized contributions by De Vahl Davis and Jones [2] and the contributors therein, many numerical methods, such as finite difference (FD), finite volume (FV), finite element (FE) and lattice Boltzmann methods, have been employed to calculate the steady, laminar flow in a square cavity [3–20]. By using the second-order finite difference scheme and combining the Richardson extrapolation method, De Vahl Davis [3] presented the benchmark solutions for the Rayleigh number up to 10^6; Hortmann et al. [4] also presented the benchmark solutions for $Ra = 10^4$–10^6 using a FV multigrid method on a fine non-uniform grid; Saitoh and Hirosel [5] obtained a fourth-order FD method using conventional five-point fourth-order approximations for the first and second derivatives, and given the benchmark solutions for $Ra = 10^4$ and 10^6; Dennis and Hudson [6] developed a h^4 compact nine-point FD scheme and obtained higher accuracy results up to $Ra = 10^5$. In their work, the governing equations were discretized by the

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fourth-order compact FD schemes, but the derivative source term was treated by the lower-order accurate scheme; Choo and Schultz [7] developed a stable fourth-order FD method to solve the steady state NS equations in the form of stream function-vorticity formulation for $Ra = 10^3 - 10^6$; Chen et al. [8] proposed a perturbational h^4 exponential FD scheme for the convective diffusion equation, and given the accuracy results for the low Rayleigh numbers $Ra = 10^3 - 10^5$; Guo et al. [9] proposed a thermal lattice Boltzmann equation for $Ra = 10^3 - 10^6$; Wang et al. [10] presented a second-order Euler–Taylor–Galerkin (ETG) FE method of fractional steps and obtained the steady solutions for $Ra = 10^4$ and 10^5. They found out that the primary flow instability, a transition from diffusive thermal conduction to a stationary time-independent steady flow structure, occurs at the critical Rayleigh number $Ra = 31304.5$, and discussed the effect of the Prandtl number on the first bifurcation; Cıbık and Kaya [11] and Benítez and Bermúdez [12] proposed a projection-based stabilization and a second order characteristics FE methods, respectively, for solving steady state natural convection problems for Rayleigh numbers up to 10^6 and 10^7; Zhang et al. [13] established a compact FD formulation on nonuniform staggered grids based on the projection method to solve the unsteady primitive variable NS equations, and given the accuracy steady results for $Ra = 10^3 - 10^5$ and discussed the process of the flow fields developing from the initial condition of zero-roll pattern for $Ra = 10^5$.

Efforts were also devoted to the problem at high Rayleigh numbers which have complicated convection patterns compared with that at the low Rayleigh numbers. Le Quéré [14] presented the benchmark solutions using a second-order Chebychev polynomial approach and Nonino and Croce [15] developed an equal-order velocity pressure FE algorithm for the Rayleigh number up to 10^8; Syrjälä [16] obtained the solutions using a higher-order Penalty–Galerkin FE approach with 8897 elements for $Ra = 10^4 - 10^7$; Tian et al. [17,18] established their fourth-order compact FD schemes for the two-dimensional (2D) steady governing equations in the form of vorticity-stream function for Rayleigh numbers up to 10^7, and proposed a FD scheme based on a stream-function-velocity formulation for $Ra = 10^3 - 10^8$. Recently Dixit and Babu [19] proposed a thermal lattice Boltzmann method to simulate natural convection in a square cavity for Rayleigh numbers $Ra = 10^3 - 10^{10}$. For the reason that the unsteadiness of the flow was not taken into account in their model, their results showed nonnegligible discrepancy with reference ones at larger values of Ra, although the results for small values were good agreement with reference ones. Bucchignani [20] described an implicit unsteady FV method for the solution of time-dependent NS equations written in terms of vorticity and velocity and provided an accurate steady solution on a fine stretched non-uniform grid at a high value of the Rayleigh number $Ra = 10^8$. Very recently Arpino et al. [21] presented the fully explicit matrix-inversion-free artificial compressibility and characteristic based split (AC–CBS) algorithm, obtained the solutions in detail the new for $Ra = 10^7$ and 10^8, and proposed a new benchmark solution for the steady-state laminar natural convection with the higher Rayleigh numbers varying from 1.24×10^8 to 1.37×10^8.

In contrast to numerical computations for lower and medium Rayleigh numbers, only a few numerical studies have been performed at very high Rayleigh numbers [22–27]. The flow structure inside the cavity with very high Rayleigh numbers becomes very complex, and the entire flow is divided into four domains, i.e., the vertical boundary layers, corner regions, boundary layers adjacent to the horizontal walls and the core region. For increasing Rayleigh number, unsteadiness sets in at a critical Rayleigh number slightly less than 2×10^8 [22,23,26], and the steady flow bifurcates to an unsteady periodic state and finally transforms to turbulent. Natural convection in cavities with high Rayleigh number is still a challenge issue because of the complex interaction between the boundary layers and the core region and the essential coupling between the transport properties of flow and thermal fields. For example, there has not any definitive explanation of the origin of the primary instability mechanism [27], the vertical boundary layer gets very thin with the increasing of the Rayleigh numbers, and so on. As a nonlinear coupled problem, the process of the nature convection includes multi-scale structures and time dependent behaviors. Even the numerical simulation of the nature convection in a simple rectangular cavity also requires a great deal of computer efforts, therefore it is important to establish efficient numerical method with high accuracy and resolution for a wide range of Rayleigh numbers and use it to study the heat transfer mechanisms especially for the sufficient large Rayleigh numbers. High-order compact FD methods, which feature high accuracy, smaller stencils and reasonable computational costs, are very popular in the computation of fluid flows and/or heat transfer (see e.g., Refs. [8,13,17,31,32,37, 28–30] and therein). In the present paper, we construct a class of at least third-order upwind compact FD schemes with free parameter (FP-UCD) and propose an optimized third-order upwind compact scheme (Opt-UCD3) based on the idea of Dispersion-Relation-Preserving (DRP). Moreover, we propose a higher-order accuracy compact FD algorithm to solve 2D unsteady NS equations in the form of stream function-vorticity formulation governing the fluid flow and heat transfer, and perform direct numerical simulations of steady, unsteady and chaotical natural convection in a air-filled square cavity with adiabatic horizontal walls for the Rayleigh numbers varying from 10^3 to 10^{10}. In addition, the unsteady features of the flow field are presented and the transition from laminar to chaotic are discussed.

The present article is organized as follows. In Section 2 the problem, the governing equations and its non-dimensionalizations are briefly recalled. The discretization of the stream function-vorticity formulation and the energy equation are explained in detail in Section 3. In Section 4 Fourier analysis and numerical experiments are performed. The unsteady solutions of natural convection are presented and discussed in Section 5. Some concluding remarks are summarized in Section 6.

2. The problem

2.1. Description of the problem

As shown in Fig. 1, the problem being solved is an incompressible Boussinesq flow filling in a 2D square cavity of width H. The homogeneous gravitational field, $\mathbf{g} = -g e_y$, is taken positive downward. In the present study, both horizontal walls are thermally insulated, while the vertical walls are isothermal, and the constant temperature of left wall θ_h is hotter than that of right wall θ_c. When the temperature difference $\Delta\theta = \theta_h - \theta_c$ between the two vertical walls exceeds a certain threshold value, natural convective motion sets in.

2.2. Governing equations

The fluid is modeled as Boussinesq fluid whose density assumed to depend linearly on temperature θ, as

$$\rho = \rho_0[1 - \beta(\theta - \theta_0)]. \quad (1)$$

where ρ_0 is the fluid density at the reference temperature θ_0, and β is the thermal expansion coefficients. The flow is described by the continuity momentum and energy equations in a Cartesian coordi-