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Generalized polynomial chaos for the convection diffusion equation with uncertainty ☆

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ABSTRACT

In this paper, several numerical algorithms are presented for solving the convection diffusion equation with random diffusivity and periodic boundary conditions. Based on the generalized polynomial chaos expansion and Galerkin projection, the stochastic convection diffusion equation is turned into a set of coupled deterministic equations. Then the implicit–explicit scheme and the fully implicit scheme are employed to temporal discretization respectively, while the Fourier spectral method is used for spatial discretization. We place emphasis on the study of the two kinds of numerical schemes with different distribution of random inputs. Numerical results show that the Uniform random inputs is special, it is that the statistical error of solution will increase rapidly after reaches the minimum as the polynomial chaos expansion growth. And the implicit–explicit scheme doesn't work well for the two-dimensional model problems. Moreover, numerical simulations by Monte Carlo method are also shown to demonstrate the efficiency and robustness of the proposed algorithms.

1. Introduction

In this paper, we consider the following convection diffusion equation with random diffusivity and periodic boundary conditions

$$u_t = \nabla \cdot [k(x;\omega)\nabla u] - v \cdot \nabla u + f(x,t;\omega), \quad (x,t;\omega) \in D \times R^+ \times \Omega,$$
$$u(x,0;\omega) = g(x;\omega), \quad x \in D \times \Omega, \tag{1}$$

where D is a bounded regular domain in $R^d(d=1,2), R^+$ is a time domain for $(0,T]$ and Ω is the sample space in an appropriately defined probability space. The concentration $u = u(x,t;\omega)$ and the source $f(x,t;\omega)$ are real-valued functions on $D \times R^+ \times \Omega$. The diffusivity $k(x;\omega)$ and initial condition $g(x;\omega)$ are real-valued functions on $D \times \Omega$, and the value of function $k(x;\omega)$ is always positive for different random inputs.

With the advance of scientist and development of society, the mathematical model we established on many real practical problems contains uncertainty drew more and more attentions. Therefore, numerical simulation of these model problem with random inputs constitutes an important research issue. Among the existing methods, a commonly used statistical approach is called Monte Carlo simulation [1,2]. In this approach, one generates realizations of random inputs based on their prescribed probability distribution. The problem becomes deterministic with each fixed realization, and by solving these deterministic problems, one collects an ensemble of solution. So the statistical information can be extracted from this ensemble. The advantage of this approach is straightforward to apply as it only requires repetitive executions of deterministic simulations. But a large number of realizations are need, for the solution statistics converge relatively slowly [12].

Another more popular non-statistical approach is called polynomial chaos expansion, use to deal with random space and obtained a set of equations without uncertainty. It is based on the homogeneous chaos theory of Wiener [3] and first applied by Ghanem and Spanos to various problems in mechanics [4–7]. It can be regarded as a spectral expansion of random variables based on the Hermite orthogonal polynomials in terms of Gaussian random variables. More recently, a more broader name called the generalized polynomial chaos [8–11], chosen from the hypergeometric polynomials of Askey scheme, is also referred as Wiener–Askey chaos [8]. Here the

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underlying random variables are chosen according to the random inputs and the weighting function of these random variables determines the type of orthogonal polynomials to be used as the basis in the random space. Such as Legendre polynomial corresponds to the Uniform distribution, Laguerre polynomial corresponds to Gamma distribution, Charlier polynomial corresponds to Poisson distribution, Krawtchouk polynomial corresponds to the binomial distribution, and so on.

According to the general process, we first apply the generalized polynomial chaos expansion to the convection diffusion equation with random diffusivity. Then following the Galerkin projection, we obtain a set of coupled deterministic equations. For the details, the reader is referred to [8,10,14]. In [11], the authors studied the two-dimensional advection–diffusion problem with random transport velocity and focused on the numerical method for random parameter space. But we place emphasis on the numerical method in time and physical space for the stochastic convection diffusion equations.In this paper, we propose the implicit–explicit (IMEX) scheme and the fully implicit (FUIM) scheme in temporal discretization respectively and the Fourier spectral method [15,16] in spatial discretization, where the IMEX scheme is a decoupled algorithm [9,10]. Note that the FUIM scheme has been used to the stochastic Allen–Cahn equation [14]. In addition, the finite difference method, the finite element method and the spectral/hp element method can also be used to physical space [10,11,13].

The rest of the paper is organized as follows. In Section 2, we describe the application of generalized polynomial chaos expansion for the stochastic convection diffusion equation. In Section 3, we apply the IMEX scheme and the FUIM scheme in temporal discretization for the deterministic equations respectively, and then employ the Fourier spectral method to obtain high accuracy in physical space. In Section 4, numerical results for the different distributions of random inputs are presented to show the efficiency of the proposed method. Moreover, numerical comparisons with the Monte Carlo simulation are reported for the two-dimensional model problems. Finally, some concluding remarks are made in the last section.

2. Application of generalized polynomial chaos expansion

In this section, we employ generalized polynomial chaos expansion to the stochastic convection diffusion equation. According to generalized polynomial chaos or the Wiener–Askey chaos was proposed in [8], we know that different orthogonal polynomials $\{\Psi_n\},\ n = 0,1,\ldots$, correspond to different random variables ξ, and all of the orthogonal polynomials satisfies the orthogonality relation. In the continuous case,

$$\langle \Psi_i(\xi), \Psi_j(\xi)\rangle = \int \Psi_i(\xi)\Psi_j(\xi)w(\xi)\mathrm{d}\xi = c_{ij}\delta_{ij}, \quad i,j = 0,1,\ldots, \tag{2}$$

and in the discrete case,

$$\langle \Psi_i(\xi), \Psi_j(\xi)\rangle = \sum_{\xi}\Psi_i(\xi)\Psi_j(\xi)w(\xi) = c_{ij}\delta_{ij}, \quad i,j = 0,1,\ldots, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the ensemble average which is the inner product in the Hilbert space of the random variables ξ and $w(\xi)$ represents weighting function, c_{ij} is nonzero constant and δ_{ij} is the Kronecker delta. For convenience of calculations and simple derivation, we standardizing the orthogonal polynomial $\{\Psi_n\}$ and using standard orthogonal polynomial $\{\Phi_n\}$ in this paper.

By applying the generalized polynomial chaos expansion, we expand the random process in Eq. (1) as

$$k(x;\omega) = \sum_{i=0}^{M} k_i(x)\Phi_i(\xi), \tag{4}$$

$$u(x,t;\omega) = \sum_{j=0}^{M} u_j(x,t)\Phi_j(\xi), \quad f(x,t;\omega) = \sum_{j=0}^{M} f_j(x,t)\Phi_j(\xi), \tag{5}$$

where the expansion coefficients $k_i(x), u_j(x,t)$ and $f_j(x,t)$ are deterministic, the value of M is determined by the dimensionality (n) of the random inputs and the highest order (p) of the polynomial expansion from

$(M+1) = (n+p)!/(n!p!)$.

Substituting the expansion (4) and (5) into Eq. (1) yields

$$\sum_{j=0}^{M}\frac{\partial u_j(x,t)}{\partial t}\Phi_j(\xi) = \nabla\cdot\left[\sum_{i=0}^{M}k_i(x)\Phi_i(\xi)\nabla\left(\sum_{j=0}^{M}u_j(x,t)\Phi_j(\xi)\right)\right] - v\cdot\sum_{j=0}^{M}\nabla u_j(x,t)\Phi_j(\xi) + \sum_{j=0}^{M}f_j(x,t)\Phi_j(\xi), \tag{6}$$

that is

$$\sum_{j=0}^{M}\frac{\partial u_j(x,t)}{\partial t}\Phi_j = \sum_{j=0}^{M}\sum_{i=0}^{M}\nabla\cdot[k_i(x)\nabla u_j(x,t)]\Phi_i\Phi_j - v \cdot\sum_{j=0}^{M}\nabla u_j(x,t)\Phi_j + \sum_{j=0}^{M}f_j(x,t)\Phi_j. \tag{7}$$

Next, a Galerkin projection of the above equation onto each polynomial basis Φ_n is then conducted in order to ensure that the error is orthogonal to the functional space spanned by the finite-dimensional basis Φ_n. By projecting with Φ_m for each $m = 0,\ldots,M$ and employing the orthogonality relation, we obtain

$$\frac{\partial u_m}{\partial t} = \sum_{j=0}^{M}\sum_{i=0}^{M}\nabla\cdot[k_i(x)\nabla u_j]e_{imj} - v\cdot\nabla u_m + f_m, \tag{8}$$

where $u_m = u_m(x,t),\ f_m = f_m(x,t),\ e_{imj} = \langle\Phi_i\Phi_m\Phi_j\rangle$, and the values of e_{imj} can be evaluated analytically from the definition of Φ_n. Setting

$$a_{mj}(x) = \sum_{i=0}^{M}k_i(x)e_{imj}, \quad b_{mj}(x) = \sum_{i=0}^{M}\nabla k_i(x)e_{imj} = \nabla a_{mj}(x), \tag{9}$$

we can rewrite the above equation as

$$\frac{\partial u_m}{\partial t} = \sum_{j=0}^{M}\nabla\cdot[a_{mj}(x)\nabla u_j] - v\cdot\nabla u_m + f_m, \tag{10}$$

or

$$\frac{\partial u_m}{\partial t} = \sum_{j=0}^{M}[a_{mj}(x)\nabla^2 u_j + b_{mj}(x)\cdot\nabla u_j] - v\cdot\nabla u_m + f_m, \quad m = 0,\ldots,M. \tag{11}$$

According to the above formulation, the stochastic convection diffusion equation is turned into a set of $(M+1)$ coupled deterministic convection diffusion equations. By using generalized polynomial chaos expansion to the initial condition in Eq. (1), we obtain

$$u(x,0;\omega) = \sum_{j=0}^{M}u_j(x,0)\Phi(\xi) = \sum_{j=0}^{M}g_j(x)\Phi(\xi) = g(x;\omega), \tag{12}$$

where the coefficient $u_j(x,0) = g_j(x)$ in the expansions is deterministic, so we obtain the initial conditions and periodic boundary conditions for each expanded equation in (11) to complete the system.

3. Numerical method and statistical error

In this section, we first consider the IMEX scheme and the FUIM scheme for Eq. (11) respectively, and then employ the Fourier