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Inverse heat conduction problem: Sensitivity coefficient insights, filter coefficients, and intrinsic verification ☆

James V. Beck [a], Keith A. Woodbury [b,*]

[a] Department of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA
[b] Mechanical Engineering Department, The University of Alabama, Tuscaloosa, AL 35487, USA

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ABSTRACT

The inverse heat conduction problem is the estimation of the time and/or space dependence of the surface heat flux or temperature utilizing interior temperature measurements at discrete times and/or locations. This problem is ill-posed since it is very sensitive to omnipresent measurement errors. Many solution methods have been proposed including exact-matching, function specification, Tikhonov regularization, iterative regularization, conjugate gradient, steepest descent and singular value decomposition. In this paper, the tools provided by the scaled sensitivity coefficients, digital filter coefficients, and intrinsic verification are used to investigate and compare several of these methods. The utility of digital filters designed for on-line instrumentation for "continuous" measurements of the surface heat flux and temperature in manufacturing settings is also demonstrated.

1. Introduction

The inverse heat conduction problem (IHCP) involves the determination of the heat flux at the surface of a solid using one or more measured internal temperature histories. A number of methods have been proposed including function specification [1,2], Tikhonov regularization (TR) [1,3–5], conjugate gradient (CG) [6–10] and singular value decomposition (SVD) methods [11–14]. The problem is challenging because it is ill-posed which is, in part, caused by the internal measurements being delayed and damped compared to the heated surface.

This paper has two main objectives. The first is to provide a method of comparison and evaluation of some available methods for the IHCP. The comparison is accomplished by providing the digital filters [15–18] underlying methods such as function specification, Tikhonov regularization, conjugate gradient, and singular value decomposition methods. Reference [8] discusses Tikhonov regularization and filter coefficients but does not provide a comparison with the other two methods. The second objective is to derive some filter coefficients that can enable a near real-time analysis; it can be helpful in real-time measurements using flame thermometers [19] and manufacturing applications. It might also have application to surface heat flux and temperature measurements in an aerospace plane or rocket re-entering the earth's atmosphere.

In order to make the task more manageable, only a single interior temperature history is considered and it is at the center of Cartesian body of thickness L. The location of interest is at $\tilde{x} = x/L = 0.5$. Two types of boundary conditions are considered at $x = L$, homogeneous in each case. The first case is for a zero temperature rise at $x = L$ with the notation X21B?0T0 and the second case is insulated at $x = L$ and denoted X22B?0T0. Filter coefficients are found for various cases, with the emphasis on the insulated remote boundary case. Related quantities are found that provide insight and contribute to intrinsic verification.

A brief explanation of the solution numbering system follows. The first character is a geometric designator, and 'X' denotes a Cartesian system. The next two digits are disignators for the boundary condition at the two interfaces ($x = 0$ and $x = L$), and the number '1' denotes a fixed temperature condition, while '2' denotes a specified heat flux condition. The letter 'B' indicates that the fields following that letter are boundary condition "modifiers" that indicate the nature of the conditions at the two boundaries just specified. The '0' indicates that the heat flux at the second boundary ($x = L$) is homogeneous. The '?' designation is introduced in this paper and supplements notation described elsewhere (see [20, pages 47–60] or [21] for more on the solution numbering system). The '?' indicates that the heat flux at the first boundary

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* Corresponding author.
E-mail addresses: beck@msu.edu (J.V. Beck), keith.woodbury@ua.edu (K.A. Woodbury).

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Nomenclature

k	thermal conductivity, W/m-K
L	thickness of 1-D domain, m
e	error
$E(\cdot)$	expected value
f	filter coefficient
$\mathbf{f}$	filter vector (a row of the filter matrix)
$\mathbf{F}$	filter matrix
$\mathbf{H}_i$	Tikhonov regularization matrix of order i, Eq. (15b,c,d)
$\mathbf{I}$	identity matrix
m_f	number of non-zero filter coefficient in the future
m_p	number of non-zero filter coefficient in the past
n_{eig}	number of eigenvalues retained in SVD method
N	total number of time steps
$\mathbf{P}$	matrix equal to $\mathbf{FX}$
q_i	heat flux component at time t_i, W/m^2
$\mathbf{q}$	vector of heat flux components, W/m^2
r	number of future time steps
R	residual
s_Y	estimated standard deviation of temperature errors, K
t	time, s
T_i	true or computed temperature at time t_i, K
$\mathbf{T}$	vector of true or computed temperatures, K
$\mathbf{w}$	descent direction
x	Cartesian coordinate, m
$\mathbf{X}$	sensitivity matrix, Eq. (3a)
Y_i	temperature measurement at time t_i, K
$\mathbf{Y}$	vector of measured temperatures, K

Greek

α	thermal diffusivity, m^2/s
α_T	Tikhonov regularization parameter
β	eigenvalue
ε_i	additive temperature error at time t_i, K
$\boldsymbol{\varepsilon}$	vector of additive temperature error, K
ϕ	temperature response due to unity heat flux, K
ρ	step size
σ	standard deviation

Subscripts/superscripts

C	a constant value
FS	function specification
f	future
k	iteration index
M	a middle value (refers to the current time step)
N	number of time steps; index of the last row of matrix
p	past
q	heat flux
s	sensor location
ss	steady-state
T	Tikhonov
T	temperature

Decorations

$\sim$	dimensionless quantity
$\wedge$	estimated value

($x = 0$) is unknown and is to be found in the solution. In fact, the '?' identifies this problem as an inverse problem. This is an important distinction from the notation described in Ref. [20] where the modifier '-' indicates an arbitrary time variation, and compound modifier 'x-' denotes an arbitrary function of space.

This paper discusses sensitivity coefficients, filter formulations and optimal comparison criteria for the inverse heat conduction problem. Based on the criteria, a comparison of the function specification, Tikhonov regularization, conjugate gradient, and singular value decomposition methods is given. Two heat flux time-variation cases are considered for each of these methods and a ranking is given based on the results.

2. Preliminary aspects

A parallel treatment of the following discussion is given elsewhere [16, see also Ref. [1], pages 10–11, 83–85, and 148–152]. The measured temperature vector $\mathbf{Y}$ which contains N elements is

$$\mathbf{Y} = [Y_1 \quad Y_2 \quad \cdots \quad Y_{N-1} \quad Y_N]^T \tag{1a}$$

The true temperature vector $\mathbf{T}$ and the additive error vector $\boldsymbol{\varepsilon}$ are

$$\mathbf{T} = [T_1 \quad T_2 \quad \cdots \quad T_{N-1} \quad T_N]^T \tag{1b}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_{N-1} \quad \varepsilon_N]^T \tag{1c}$$

Each of the above elements in Eqs. (1a, b, c) are at the same location; the subscripts indicate the running times. The measured temperature vector, true temperature vector and error vector are related in the additive manner

$$\mathbf{Y} = \mathbf{T} + \varepsilon \tag{2a}$$

The true, but unknown, surface heat flux vector is similarly given by

$$\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_{N-1} \quad q_N]^T \tag{2b}$$

In this paper the heat flux is approximated by piecewise constants. In the linear IHCP the true temperature vector $\mathbf{T}$ is related to the heat flux vector $\mathbf{q}$ by the relation

$$\mathbf{T} = \mathbf{Xq} \tag{2c}$$

where $\mathbf{X}$ is a sensitivity matrix; it is a square matrix with N by N elements and is given by

$$\mathbf{X} = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_1 & 0 & \cdots & 0 & 0 \\ X_3 & X_2 & X_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ X_{N-1} & X_{N-2} & X_{N-3} & \cdots & X_1 & 0 \\ X_N & X_{N-1} & X_{N-2} & \cdots & X_2 & X_1 \end{bmatrix} \tag{3a}$$

Notice that this square matrix has the same elements in each column but each successive column is shifted down one. The numerical values for the first column of Eq. (3a) can be found by using a heat flux of unity value. For the approximation of piecewise constant values for the heat flux history, the elements of the sensitivity matrix are given by

$$X_1 = \phi_1, \; X_2 = \phi_2 - \phi_1, \ldots, \; X_N = \phi_N - \phi_{N-1} \tag{3b}$$

where ϕ_n is the dimensionless temperature at location $\tilde{x}$ for a unit surface heat flux at times $t_1 = \Delta t$, $t_2 = 2\Delta t, \ldots$, $t_N = N\Delta t$ or in dimensionless form as, $\tilde{t}_1 = \alpha\Delta t/L^2, \ldots, \; \tilde{t}_N = \alpha N\Delta t/L^2$.

The structure of $\mathbf{X}$ with the zeros in the upper right is an indication of causality which implies that the computed temperature at a time t_M is a function only of q_M and previous components. More specifically, the computed temperature at time t_M is not a function of the future components $q_{M+1}, \ldots, q_N$. Although this