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On defects of Taylor series approximation in heat conduction models

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ABSTRACT

Taylor series approximation like $q(t+\tau) \approx q + \tau\frac{\partial q}{\partial t}$ are often used to derive, extend or interpret typical heat conduction models. Researchers may take it for granted that the single-phase-lagging (SPL) model can be considered as an extension of the Cattaneo–Vernotte (CV) model because there is such approximation relationship between them. We point out in this paper that this Taylor series approximation itself has some defects based on analyses in mathematics, physics and some examples first. Then, we show essential differences in both mathematics and physics between the CV and SPL models. It is found that their mathematical characteristics and accordance with the laws of thermodynamics are significantly different, which indicates that using this approximation to connect the two models may be defective in some cases. What's more, higher order approximation can't solve these problems and defects.

1. Introduction

Fourier's law of heat conduction is often used to describe normal heat conduction problems in engineering. In recent years, the limitations of Fourier's law have been revealed [1–6] that Fourier's law predicts an unphysically infinite speed of heat perturbation propagation and it fails to characterize supertransient and high heat flux processes well. Several modified heat conduction models were proposed to get over these limitations. The Cattaneo–Vernotte (CV) model [7,8] is the most typical one which leads to hyperbolic heat conduction equation and wave-like transport in heat conduction processes, called thermal wave. Jeffrey model [2] can be considered as an extension of the CV model since it takes into account the influence of temperature relaxation. Tzou [9] proposed the single-phase-lagging (SPL) model which can reduce to the CV model by taking first-order Taylor series approximation. Anisinov et al. [10] proposed a model for metals by regarding the interactions of electron and phonon. Guyer et al. [11] developed a representative model for pure phonon heat conduction. There are also further modifications and improvements of these classical models. Tzou [12] proposed a dual-phase-lagging model to add the influence of temperature lag on the basis of the single-phase-lagging model. Coleman et al. [13] improved the changing rate of the heat energy. Most of these models are linear and predict limited heat conduction speed, getting over the infinite speed problem in Fourier's law. There are also some non-linear models which predict limited heat conduction speed. Thermomass theory [14–17] for heat conduction under extreme conditions is just one of them based on relativity and mass-energy equation. Alternative approaches to the analysis of the diffusion equation [18–20] is another non-linear model whose equation can be changed to Burger's equation and therefore, some existing conclusions in math can be used to analyze heat conduction problems.

We have seen from the above brief review that Taylor series approximation is adopted in several heat conduction models. Here the CV and SPL models are taken as typical examples. The CV model is expressed as

$$q + \tau\frac{\partial q}{\partial t} + \lambda\nabla T = 0, \tag{1}$$

where τ is the thermal relaxation time, q is the heat flux density, λ is the thermal conductivity and T is the temperature. The CV model is used to describe the supertransient heat conduction and also agrees well with some of experiments and simulations. Consider the single-phase-lagging model [9]

$$q(x,y,z,t+\tau) + \lambda\nabla T = 0. \tag{2}$$

Comparing Eq. (2) with the CV model Eq. (1), we find that for $q(x,y,z,t+\tau)$, if we use first-order Taylor series approximation

$$q(t+\tau) \approx q + \tau\frac{\partial q}{\partial t}. \tag{3}$$

Eq. (2) will reduce to Eq. (1). Because of this approximation relationship between them, the SPL model is considered as an extension or explanation of the CV model and similar approximation methods, such as temperature Taylor series approximation, are

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also used in deriving other typical heat conduction models [21–25]. This approximation method is assumed to cause negligibly small influence because the relaxation time is very small.

In this paper, however, we note that the Taylor series approximation can lead to very large difference no matter how small the relaxation time is. Even if the relaxation time is very small, the deviation between the two sides of Eq. (3) can be very large, and there are also some essential differences in mathematics and physics between the CV and SPL models. Their mathematical characteristics and accordance of the laws of thermodynamics are very different, which shows that using this approximation to connect the two models is defective. In addition, higher order approximation can't solve the mathematical and physical problems caused by first-order approximation.

2. Influence of Taylor series approximation

2.1. Deviation of heat flux field

In the approximation of heat flux Eq. (3), $q(t+\tau)$ doesn't equal to $q+\tau\frac{\partial q}{\partial t}$ strictly. Therefore, this approach is considered as a special Taylor series approximation when the relaxation time τ is very small. But in fact, the deviation between the two sides of Eq. (3) is uncertain in mathematics. First, not all functions have Taylor series, even infinitely differentiable functions. For these functions which don't have Taylor series, Taylor series approximation is infeasible because the remainders don't tend to zero. Therefore the deviation between the Taylor series and function is not sure. In addition, Taylor series approximation is feasible only in their convergence regions. That is to say, even if a function has Taylor series, the approximation only exists in some certain regions. Second, even if we can make sure that Taylor series approximation exists and the relaxation time τ is very small, the deviation between the two sides of Eq. (3) is not necessarily very small. This is because the relaxation time τ is a physical property. It must be a real number, not an "infinitesimal" in mathematics. As long as τ is not an infinitesimal, the deviation between $q(t+\tau)$ and $q+\tau\frac{\partial q}{\partial t}$ can still be very large because the higher order derivative terms $\tau^n\frac{\partial^n q}{\partial t^n}$ are unknown. Although τ^n are very small, $\frac{\partial^n q}{\partial t^n}$ can also be very large and, their products are uncertain. Because of these uncertain higher order derivative terms, the deviation between $q(t+\tau)$ and $q+\tau\frac{\partial q}{\partial t}$ is uncertain either. In summary, even if the relaxation time τ is very small, the deviation between the two sides of Eq. (3) can still be very large. As examples, we will discuss this problem below in some common functions which often appear in heat conduction problems. Consider the heat conduction equations of Fourier's law Eq. (4) and the CV model Eq. (5)

$$\frac{\partial T}{\partial t}=\frac{\lambda}{\rho c_V}\nabla^2 T, \tag{4}$$

$$\frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2}=\frac{\lambda}{\rho c_V}\nabla^2 T. \tag{5}$$

For Eq. (4), a general method is to make a separation of variables $T=f(t)g(x)$. Substituting it into Eq. (4) gives

$$\frac{f'}{f}=\frac{g''}{g}=-\lambda_n, \tag{6}$$

$$f'(t)+\lambda_n f(t)=0. \tag{7}$$

Solving this ordinary differential equation, we obtain $f(t)=Ce^{-\lambda_n t}$. The part determined by time of temperature field has a form of exponential function. For Eq. (5), we can also make a separation of variables $T=f(t)g(x)$. Substituting it into Eq. (5) gives

$$\frac{\frac{1}{\tau}f'+f''}{f}=\frac{\lambda}{\rho c_V\tau}\frac{g''}{g}=-\lambda_n, \tag{8}$$

$$\frac{1}{\tau}f'+f''+\lambda_n f=0. \tag{9}$$

There will be different cases. When $\frac{1}{\tau^2}-4\lambda_n>0$, the solution is $f(t)=A_1e^{x_1 t}+B_1e^{x_2 t}$ which also has a form of exponential function. $x_1,\ x_2$ are the real roots of $x^2+\frac{1}{\tau}x+\lambda_n=0$. When $\frac{1}{\tau^2}-4\lambda_n<0$, the solution is $f(t)=e^{x_3 t}(A_2\sin x_4 t+B_2\cos x_4 t)$. $x_3+x_4 i,\ x_3-x_4 i$ are the complex roots of $x^2+\frac{1}{\tau}x+\lambda_n=0$. We can find that not only exponential function but also trigonometric function appears. From the above analyses we can find that the part of temperature field determined by time can be expressed by exponential and trigonometric functions in the method of separation of variables. So, we can make sure the deviation between the two sides of Eq. (3) in these functions to show this deviation in heat conduction problems.

2.1.1. Deviation in trigonometric function

Consider a heat conduction problem with heat source $\phi=-\frac{2n\pi q_0\rho c_V x}{\lambda\tau}\cos\frac{2n\pi t}{\tau}$. In this case, the energy conservation equation is expressed as

$$\frac{\partial q}{\partial x}=-\rho c_V\frac{\partial T}{\partial t}+\phi. \tag{10}$$

Substituting it into Eq. (2) leads to

$$\frac{\rho c_V}{\lambda}\frac{\partial q(t+\tau)}{\partial t}+\frac{\partial\phi}{\partial x}=\nabla^2 q. \tag{11}$$

The initial condition is taken $q|_{t=0}=0$ and the boundary conditions are taken $q|_{x=0,l}=q_0\sin\frac{2n\pi t}{\tau}$. For this problem, we can get its classical solution

$$q(x,t)=q_0\sin\frac{2n\pi t}{\tau}. \tag{12}$$

It is worth mentioning that for this problem, Eq. (12) is also equivalent to Fourier's Law because $q(x,t)=q(x,t+\tau)$. Then we can use Eq. (12) to show the deviation in Eq. (3). For the heat flux expressed by Eq. (12), we can get its Taylor series approximation

$$q+\tau\frac{\partial q}{\partial t}=q_0\left(\sin\frac{2n\pi t}{\tau}+2n\pi\cos\frac{2n\pi t}{\tau}\right). \tag{13}$$

The relative deviation between $q(t+\tau)$ and $q+\tau\frac{\partial q}{\partial t}$ is

$$\eta=\frac{q+\tau\frac{\partial q}{\partial t}-q(t+\tau)}{q(t+\tau)}=2n\pi\cot\frac{2n\pi t}{\tau}. \tag{14}$$

We find that Eq. (14) is a periodic function and its value can reach infinity. No matter how small the relaxation time τ is (larger than zero), the relative deviation can still be very large. The large deviation will always appear because Eq. (14) is a periodic function. Fig. 1 shows the heat flux fields with the form of trigonometric function which belongs to original heat flux $q(t+\tau)$ and heat flux with Taylor approximation $q+\tau\frac{\partial q}{\partial t}$ (q is expressed by Eq. (12) and $n=1$). In Fig. 1, the heat flux with Taylor approximation has far larger amplitude than the original heat flux. Therefore, the deviation caused by Taylor approximation can be very large and we find that the difference between the two heat flux fields is in periodical vibration. In fact, Fig. 1 is for the case of $n=1$, and the difference between them will be larger and larger with the increase of n.