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Exact solution of Guyer–Krumhansl type heat equation by operational method

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ABSTRACT

We construct particular solutions for some heat transport differential equations, in particular, for extended forms of hyperbolic heat equation and of Guyer–Krumhansl (GK) equation. The operational approach, integral transforms, generalized orthogonal polynomials and special functions are used. Examples of heat propagation in non-Fourier models are studied and compared with each other. Analytical solutions for some three-dimensional heat transport equations are obtained. The exact analytical solutions for GK type heat equation with linear term are derived. The description of an instant heat surge propagation and of power-exponential pulse is given in heat transport models of Fourier, Cattaneo and Guyer–Krumhansl. Space–time propagation of a periodic function, obeying telegraph and GK equations with linear terms is studied by the operational technique. The exact bounded analytical solutions are obtained. The role of various terms in the equations is illustrated and their influence on the solutions is elucidated. The application for ballistic heat flow study with account for Knudsen number is provided.

1. Introduction

Fourier's law of heat conduction, which relates linearly the temperature gradient (the cause) to the heat flux (the effect) is one of the most popular laws in continuum physics as it provides an excellent agreement between theory and experiment for more than 90% of the cases. It is undoubtedly the best model for heat conduction in undeformable solids. Fourier's law of heat conduction is one of the most important laws of physics in our everyday life. It constitutes the best model for heat conduction in many solids, relating linearly the temperature gradient to the resulting heat flux However, it has some shortcomings. As noted by L. Onsager in 1931, the Fourier's model was in contradiction with the principle of microscopic reversibility [1]. He writes that this contradiction "is removed when we recognize that is only an approximate description of the process of conduction, neglecting the time needed for acceleration of the heat flow". Despite its success, the Fourier fails to describe heat conduction at low temperature <25 K, in particular, in dielectric crystals, and in small systems. Fourier law displays the unphysical properties; in particular, it lacks inertial effects: if an instant temperature perturbation is applied at a point in the solid, it will be felt instantaneously and everywhere at all distant points. The most important related phenomenon is the so-called second sound, when temperature disturbance propagates like damped waves. To overreach the problems associated with Fourier's law, Cattaneo [2] proposed a time-dependent relaxational model, which yielded the following equation:

$$(\tau\partial_t^2 + \partial_t)T = k_T\nabla^2 T, \tag{1}$$

where τ is an intrinsic thermal property of the media, characterizing the time needed for the initiation of a heat flow after a temperature gradient appears at the boundary of the domain, and k_T denotes heat diffusivity. The time τ is often related to the speed of the second sound C in media ($\tau = k_T/C^2$); $\sqrt{k_T/\tau} = C$ represents a velocity like quantity, associated with the speed of the heat wave in the medium, which characterizes the thermal wave propagation the same way as the diffusion behavior is characterized by the diffusivity. The relaxation time τ in heat conduction is extremely small ($\tau \approx 10^{-13}$ s) at room temperature. Eq. (1) is the simplest model of the second sound phenomenon observed first in liquid Helium [3]. Later on, the analysis of the theoretical background [4] resulted in the observation of second sound also in solid crystals [5], via properly designed experiments [6–8]. In these tests the heat pulse technology was crucial for the sensitive detection of the thermal diffusivity.

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However, the hyperbolic heat equation does not properly describe heat conduction. Numerous mathematical contradictions, related to this equation, were evidenced (see, for example [9–11]); for example, it was shown that the telegraph equation did not preserve the non-negativity of its solutions and that the maximum principle was not valid for the hyperbolic heat equation even in the one-dimensional space. Moreover, some physical contradictions, related to Cattaneo's equation, were also noted (see, for example, [12,13]). It was shown that experimental data did not confirm what this equation predicted. Thus, Cattaneo's equation for heat propagation was superseded by others. Some of the advances are described in [14].

The spectrum of mathematical studies of heat-type equations and their solutions is extremely broad. In what follows we propose operational approach to their solution [15–17]. Its evident advantage is that it allows exact and straightforward analytical solutions, when combined with operational presentation of generalized forms of Hermite, Laguerre and other orthogonal polynomials [18–20], covering also high order equations [21,22]. Moreover, many of these equations describe different physical phenomena in other segments of physics. Recently evolution of Gauss and Airy packets, governed by the one-dimensional Schrödinger equation for a charge in constant electric field

$$i\partial_\tau \Psi(x,\tau) = -\partial_x^2 \Psi(x,\tau) + bx\Psi(x,\tau) \tag{2}$$

was studied in [16] by the operational method [15]. In this context we recall that particle propagation under a potential barrier obeys Schrödinger equation, subject to $t \to i\tau$ change. Then, upon denoting $\Psi \to F$, Eq. (3) becomes

$$\partial_t F(x,t) = \alpha\partial_x^2 F(x,t) + \beta x F(x,t), \tag{3}$$

which is Fourier heat equation with linear coordinate term, this latter being non-essential in the context of heat transport. The above Eq. (3) does make sense in the imaginary time formalism, i.e. in the Euclidean picture in quantum mechanics (see, e.g. [23]). This equation is important in quantum chromo dynamics vacuum and describes tunneling of a particle through a region, where potential energy is greater than the particle energy. The operational solution of Eq. (3) can be expressed also via Gauss integral and it reads as follows [16]:

$$F(x,t) = e^{\Phi(x,t;\alpha,\beta)}\hat{\Theta}\hat{S}f(x) = e^{\Phi(x,t;\alpha,\beta)}\frac{1}{2\sqrt{\pi\alpha t}}\int_{-\infty}^{\infty} e^{-\frac{(x+\alpha\beta t^2-\xi)^2}{4t\alpha}} f(\xi)d\xi, \tag{4}$$

where $\Phi(x,t;\alpha,\beta) = \frac{1}{3}\alpha\beta^2 t^3 + \beta t x$ is the phase, $\hat{\Theta} = e^{\alpha\beta t^2 \partial_x}$ is the translation operator and the heat diffusion operator

$$\hat{S} = \exp(\alpha t \partial_x^2) \tag{5}$$

was thoroughly explored by Srivastava in [24]. Evidently, for heat transport with $\beta = 0$ we get $\Phi = 0$. Note that the differential operator in the exponential (5) reduces to the first order derivative with the help of the following integral presentation for the exponential of a square of an operator $\hat{p}$ [25]:

$$\exp(\hat{p}^2) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp(-\xi^2 + 2\xi\hat{p})d\xi, \tag{6}$$

where $\hat{p} = \sqrt{\alpha t}D$. Translation operator $\hat{\Theta}$ produces shift: $\exp^{\eta(D+\alpha)} f(x) = \exp^{\eta\alpha} f(x+\eta)$. The solution (4) consists of the action of the evolution operator on the initial condition $F(x,0) = f(x)$, which is transformed by $\hat{S}$ and $\hat{\Theta}$. While it is not always possible to compute the result of the operatorial action of $\hat{S}$ and $\hat{\Theta}$ on arbitrary function, it is easy to obtain the result for the initial monomial $f(x) = x^n$. The action of the heat diffusion operator on it simply yields the Hermite polynomials, defined below. Moreover, we can choose more general initial condition function $f(x) = x^k e^{\delta x}$ and according to the operational rule

$$\exp(yD_x^2)x^k e^{\alpha x} = e^{(\alpha x + \alpha^2 y)} H_k(x + 2\alpha y, y), \tag{7}$$

where the Hermite polynomials of two variables $H_n(x,y)$ are defined as follows [26]:

$$H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}} x^n = n!\sum_{r=0}^{[n/2]} \frac{x^{n-2r}y^r}{(n-2r)!r!}, \quad H_n(x,y) = (-i)^n y^{n/2} H_n\left(\frac{ix}{2\sqrt{y}}\right) \tag{8}$$

we obtain $\hat{S}f(x) = \exp(\delta(x+\delta a))H_k(x+2\delta a, a) = f(x,t)$, $a = \alpha t$. The consequent action of the translation operator $\hat{\Theta}$ yields the shift along the x argument and results in

$$F(x,t) = e^{\Phi+\Delta_1} H_k(x + 2t\alpha\delta + t^2\alpha\beta, \alpha t), \tag{9}$$

where $\Delta_1 = \delta(x + \delta\alpha t + \alpha\beta t^2)$. For heat conduction we can assume $\beta = 0$ in (3) and then we get the following simple solution:

$$F(x,t)|_{\beta\to 0 f(x)=x^k e^{\delta x}} = H_k(x + 2t\alpha\delta, \alpha t)\exp\{\delta(x+\delta\alpha t)\} \tag{10}$$

For $\delta = 0$ we immediately obtain the result for $f(x) = x^n$:

$$F(x,t)|_{\beta\to 0} = e^{\Phi} H_n(x + \alpha\beta t^2, \alpha t)\big|_{\beta\to 0} = H_n(x,\alpha t). \tag{11}$$

This result is useful also if the initial function is expandable in series $f(x) = \sum_n c_n x^n$ or can be approximated by them; then the solution appears in the form of series too: $F(x,t) = e^{\Phi}\sum_n c_n H_n(x + \alpha\beta t^2, \alpha t)$. Moreover, the initial function $f(x) = x^k e^{\delta x}$ is more general and itself represents a surge with power rise and common exponential fade for $\delta < 0$ and positive coordinate $x > 0$ (i.e. $x^2 e^{-x}$, etc.). It will be discussed in what follows.

Consider the following two-dimensional heat propagation equation with the linear terms:

$$\partial_t F(x,y,t) = \left\{\left(\alpha\partial_x^2 + \beta\partial_x\partial_y + \gamma\partial_y^2\right) + bx + cy\right\}F(x,y,t), \quad \min(\alpha,\beta,\gamma) > 0 \tag{12}$$

and the initial condition $F(x,y,0) = f(x,y)$. It can be solved by the operational method, which yields the following two-dimensional generalization of the solution (4):

$$F(x,y,t) = e^{\Psi}\hat{\Theta}_x\hat{\Theta}_y\hat{E}f(x,y), \tag{13}$$

where $\hat{\Theta}_x = e^{t^2(\alpha b + \beta c/2)\partial_x}$ and $\hat{\Theta}_y = e^{t^2(\gamma c + \beta b/2)\partial_y}$ are the translation operators for each of the two coordinates, $\Psi = (\alpha b^2 + \gamma c^2 + \beta bc)t^3/3 + t(bx + cy)$ is the phase and

$$\hat{E} = \exp\left[t\left(\alpha\partial_x^2 + \beta\partial_x\partial_y + \gamma\partial_y^2\right)\right] \tag{14}$$

represents the heat diffusion operator for the two-dimensional case, analogous to $\hat{S}$ operator (5) in the one-dimensional case. The result of the action of the heat diffusion operator $\hat{E}$ on the initial condition $f(x,y)$ is $f(x,y,t) = \hat{E}f(x,y)$ and, consequently, the commuting diffusion operators $\hat{\Theta}_x\hat{\Theta}_y$ shift the argument of the function f. In complete analogy with the one-dimensional case we obtain the solution of the two-dimensional heat conduction equation with lineal terms (12) in the following form:

$$F(x,y,t) = e^{\Psi}\hat{\Theta}_x\hat{\Theta}_y\hat{E}f(x,y) \propto f\left(x + t^2(\alpha b + \beta c/2), y + t^2(\gamma c + \beta b/2), t\right). \tag{15}$$

The explicit double integral form of the operator $\hat{E}$ was obtained in [20]; it is the Gauss type integral, which we omit here for the sake of conciseness. In the case of higher space dimensions the heat diffusion is executed by one-dimensional operators (5) $\hat{S}_x\hat{S}_y$. The solution in this case reads as follows: