International Journal of Heat and Mass Transfer 94 (2016) 449–464

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

An invariant method of fundamental solutions for two-dimensional steady-state anisotropic heat conduction problems

Liviu Marin *

Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 14 Academiei, 010014 Bucharest, Romania
Institute of Solid Mechanics, Romanian Academy, 15 Constantin Mille, 010141 Bucharest, Romania

ARTICLE INFO

Article history:
Received 30 August 2015
Received in revised form 21 November 2015
Accepted 21 November 2015
Available online 11 December 2015

Keywords:
Anisotropic heat conduction
Laplace–Beltrami's equation
Method of fundamental solutions (MFS)
Invariant method of fundamental solutions (IMFS)
Exact and noisy data
Regularization

ABSTRACT

We investigate both theoretically and numerically the so-called invariance property, see e.g. Sun and Ma (2015a,b), of the solution of boundary value problems associated with the anisotropic heat conduction equation (or Laplace–Beltrami's equation) in two dimensions with respect to elementary transformations of the solution domain, e.g. dilations or contractions. We also show that the standard method of fundamental solutions (MFS) does not satisfy the invariance property. Motivated by these reasons, we introduce, in a natural manner, a modified version of the MFS that remains invariant under elementary transformations of the solution domain and is referred to as the invariant MFS (IMFS). Five two-dimensional examples are thoroughly investigated to assess the numerical accuracy, convergence and stability of the proposed IMFS, in conjunction with the Tikhonov regularization method (Tikhonov and Arsenin, 1986) and Morozov's discrepancy principle (Morozov, 1966), for Laplace–Beltrami's equation with perturbed boundary conditions.

1. Introduction

Numerous natural and man-made materials cannot be considered isotropic and the dependence of the thermal conductivity with direction has to be taken into account in the modelling of the heat transfer. More specifically, crystals, wood, sedimentary rocks, metals that have undergone heavy cold pressing, laminated sheets, composites, cables, heat shielding materials for space vehicles, fibre reinforced structures, and many others are examples of anisotropic materials. Composites are also of special interest to the aerospace industry because of their strength and reduced weight. Consequently, heat conduction in anisotropic materials has numerous important applications in various branches of science and engineering and hence its understanding is of great importance. The mathematical problems associated with anisotropic heat conduction equation, also referred to as Laplace–Beltrami's equation, have been the subject of numerous studies using various numerical methods, e.g. the finite-difference method (FDM) [1–3], the boundary element method (BEM) [4–6], the finite element method (FEM) [2,7,8], the method of fundamental solutions (MFS) [9–12], the singular boundary method (SBM) [13] etc.

The MFS is a meshless boundary collocation method which belongs to the family of so-called Trefftz methods [14,15] and is applicable to BVPs in which a fundamental solution of the operator in the governing equation is known. Despite this restriction, the MFS has become very popular primarily because of the ease with which it can be implemented, particularly for the problems in complex geometries. The MFS was originally proposed by Kupradze and Aleksidze [16] and later introduced as a numerical method by Mathon and Johnston [17]. Since then, it has been successfully applied to a large variety of physical problems, an account of which may be found in the survey papers [9,18–20]. However, there exist some heat conduction problems for which the simple application of the MFS is not sufficient to obtain an accurate numerical solution, e.g. problems related to domains containing a boundary singularity generated by the presence of a crack or a V-notch, and hence the standard MFS has to be modified/enriched.

In the case of isotropic heat conduction problems (i.e. boundary value problems for Laplace's equation), Alves and Leitao [21] proposed an enriched MFS to simulate the presence of a crack, whilst Marin [22] employed the MFS in conjunction with the corresponding singular solutions as given by the asymptotic expansion of the solution near the singular point. Saavedra and Power [23,24] added a constant to the standard MFS approximation for isotropic heat conduction problems, at the same time mentioning that this

* Address: Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 14 Academiei, 010014 Bucharest, Romania and Institute of Solid Mechanics, Romanian Academy, 15 Constantin Mille, 010141 Bucharest, Romania.

E-mail addresses: marin.liviu@gmail.com, liviu.marin@fmi.unibuc.ro

http://dx.doi.org/10.1016/j.ijheatmasstransfer.2015.11.080
0017-9310/

constant is necessary to be added in particular for two-dimensional problems because of completeness issues. Chen et al. [25] also modified the standard MFS formulation for both interior isotropic heat conduction problems with a degenerate-scale domain and exterior problems for Laplace's equation with a bounded solution at infinity by adding a constant together with a constraint. This enrichment used in [25] is required for obtaining a unique solution of the problems considered.

Recently, Sun and Ma [26,27] proposed a modified MFS for isotropic heat conduction problems in two dimensions, referred to as the invariant MFS (IMFS), which also preserves the invariance property as the exact solution of the problem undergoes elementary transformations, e.g. dilations, contractions. Therefore, it is the purpose of this paper to extend the IMFS proposed in [26,27] to anisotropic heat conduction problems with exact and noisy boundary conditions, as well as implement and investigate the performance of the IMFS for such problems.

The paper is organised as follows: In Section 2 we formulate mathematically the problem under investigation. The invariance property of the solution of anisotropic heat conduction problems with respect to dilations/contractions is introduced and proved in Section 3. Section 4 is devoted to the brief description of the standard MFS and the introduction of its modified version that also satisfies the invariance property. The Tikhonov regularization method, as well as Morozov's discrepancy principle for selecting an appropriate regularization parameter, are briefly described in Section 5. The accuracy, convergence and stability of the numerical results obtained using the proposed IMFS are thoroughly analysed for five two-dimensional examples in Section 6. Finally, some concluding remarks are made in Section 7.

2. Mathematical formulation

Consider a bounded connected domain $\Omega \subset \mathbb{R}^2$ occupied by an anisotropic solid characterised by the homogeneous, symmetric and positive-definite thermal conductivity tensor $\mathbf{K} = [K_{ij}]_{1\leqslant i,j\leqslant 2} \in \mathbb{R}^{2\times 2}$, i.e.

$$\mathbf{K}^{\top} = \mathbf{K}; \tag{1a}$$

$$\xi \cdot \mathbf{K}\xi \geqslant 0, \quad \forall \xi \in \mathbb{R}^2; \quad \xi \cdot \mathbf{K}\xi = 0 \Longleftrightarrow \xi = \mathbf{0}. \tag{1b}$$

We also assume that Ω is bounded by a smooth or piecewise smooth curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \varnothing, \Gamma_2 \neq \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$.

In the absence of heat sources, the temperature distribution, u, in the domain Ω satisfies the following elliptic partial differential equation, also referred to as the anisotropic heat conduction equation, see e.g. [2],

$$-\nabla \cdot (\mathbf{K}\nabla u(\mathbf{x})) \equiv -\sum_{i,j=1}^{2} K_{ij}\partial_i\partial_j u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{2}$$

where $\partial_j \equiv \partial/\partial x_j$. We now let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector at $\mathbf{x} \in \partial\Omega$ and $q(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ defined by, see e.g. [2],

$$q(\mathbf{x}) = -\mathbf{n}(\mathbf{x}) \cdot (\mathbf{K}\nabla u(\mathbf{x})) \equiv -\sum_{i,j=1}^{2} n_i(\mathbf{x})K_{ij}\partial_j u(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

In the direct problem formulation, the anisotropic heat conduction Eq. (2) is solved together with appropriate Dirichlet, Neumann, or Robin boundary conditions to obtain the temperature distribution in the solution domain Ω, as well as the corresponding unknown boundary conditions. More specifically, herein we assume the following boundary conditions attached to Eq. (2):

(a) Dirichlet boundary conditions (i.e. prescribed temperature)

$$u(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega; \tag{4a}$$

(b) Neumann boundary conditions (i.e. prescribed normal heat flux)

$$q(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega; \tag{4b}$$

(c) mixed boundary conditions (i.e. prescribed temperature and normal heat flux)

$$u(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1; \qquad \text{and} \qquad q(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2; \tag{4c}$$

where $\widetilde{u}$ is the prescribed boundary temperature on $\partial\Omega$ or Γ_1 according to case (a) or (c), whilst $\widetilde{q}$ is the prescribed normal heat flux on $\partial\Omega$ or Γ_2 according to case (b) or (c).

We have decided not to investigate Robin boundary conditions herein since the latter represent a linear combination of Dirichlet and Neumann boundary conditions prescribed on the same segment of the boundary and, consequently, they do not bring any additional information to the present study. Moreover, we also assume in this paper that the boundary conditions related to cases (a)–(c) and associated with the anisotropic heat conduction problem (2) are noisy and this corresponds to real life problems encountered in practice.

3. Invariance property of solution

Without any loss of the generality, we further consider the Dirichlet problem associated with the anisotropic heat conduction equation in a simply connected domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$ given by Eqs. (2) and (4a), namely

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla u(\mathbf{x})) = 0, & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = \widetilde{u}(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases} \tag{5}$$

It is well-known that problem (5) has a unique solution $u \in \mathrm{H}^1(\Omega)$, provided that $\widetilde{u} \in \mathrm{H}^{1/2}(\partial\Omega)$, see e.g. [28], and this unique solution of problem (5) admits the following double-layer representation in the solution domain, see e.g. [29,30]:

$$u(\mathbf{x}) = \int_{\partial\Omega} \left[\mathbf{n}(\mathbf{y}) \cdot \left(\mathbf{K}\nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})\right)\right] \varphi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}), \quad \mathbf{x} \in \Omega, \tag{6}$$

where $\varphi \in \mathrm{H}^{\alpha}(\partial\Omega)$ is a charge/surface density and G is a fundamental solution of the anisotropic heat conduction Eq. (2). Here, $\alpha \geqslant 0$ and we make the convention $\mathrm{H}^0(\partial\Omega) \equiv \mathrm{L}^2(\partial\Omega)$.

By employing the jump conditions for representation (6) and the Dirichlet boundary condition associated with problem (5), one obtains the following double-layer representation on the boundary of the solution domain, see e.g. [29,30]:

$$\int_{\partial\Omega} \left[\mathbf{n}(\mathbf{y}) \cdot \left(\mathbf{K}\nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})\right)\right] \varphi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) - \frac{1}{2}\, \varphi(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{7}$$

We note that relation (7) actually represents a boundary integral equation for determining the unknown charge/surface density, φ.

Consider the operator defined by

$$\mathcal{K} : \mathrm{H}^{\alpha}(\partial\Omega) \longrightarrow \mathrm{H}^{\alpha}(\partial\Omega), \quad \varphi \in \mathrm{H}^{\alpha}(\partial\Omega) \mapsto \mathcal{K}\varphi \in \mathrm{H}^{\alpha}(\partial\Omega), \tag{8a}$$

where $\alpha \geqslant 0$ and

$$(\mathcal{K}\varphi)(\cdot) : \partial\Omega \longrightarrow \mathbb{R},$$

$$(\mathcal{K}\varphi)(\mathbf{x}) := \int_{\partial\Omega} \left[\mathbf{n}(\mathbf{y}) \cdot \left(\mathbf{K}\nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})\right)\right] \varphi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega. \tag{8b}$$