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Heat transfer in fractal materials

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ABSTRACT

Heat transfer in fractal materials is considered in the framework of continuous models with non-integer dimensional spaces. We use a recently proposed vector calculus in non-integer dimensional spaces to describe heat flow in fractal materials. Solutions of the steady heat flow in fractal pipe and rod are derived.

1. Introduction

A main characteristic of fractal materials is non-integer physical dimensions such "particle" and mass dimensions [1,2]. A description of heat transfer in fractal media is important to determine physical properties of materials [3–9]. Fractal media can be described in the framework of continuous models are based on the concept of density of states of power-law type [2]. Continuous models for fractal distributions of particle, media and fields have been proposed in [10,11,2], and then these models have been developed by Ostoja-Starzewski [12–14] and other scientists. The continuous models of fractal materials can use tools of integration and differentiation for a non-integer dimensional space [21–23], which were recently developed in [15,16].

The integration over non-integer dimensional spaces (NIDS) has a wide application in quantum field theory [21] for the dimensional regularization of ultraviolet divergences. A generalization of integration for NIDS is proposed by Stillinger in [22], and then it has been generalized by Palmer and Stavrinou [23] by using product measure method. The scalar Laplace operators for NIDS also has been proposed in [22,23]. Papers [22,23] consider only the scalar Laplacian for NIDS. The first order NIDS-operators such as gradient, divergence, curl operators and the vector Laplacian [24] are not considered in [22,23]. A possibility to use only the scalar Laplacian greatly restricts us in application of continuous models for fractal materials. Recently, a vector calculus for NIDS, where the first and second orders differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for NIDS, have been proposed in [15,16]. The suggested calculus allows us to describe fractal materials, for which the volume dimension D of the material region and the dimension d of boundary of this region are not related by the equation $d = D - 1$. The suggested NIDS vector calculus has been used to describe fractal media by continuous models in the elasticity theory of fractal material [17], the fractal electrodynamics [18,19], and the fractal hydrodynamics [20]. In this paper, we consider the heat flow in fractal pipe and rod. We solve the corresponding heat transfer equation for fractal material in the general isotropic case, where the condition $d = D - 1$ for volume ("D") and boundary (d) fractal dimensions is not used.

2. Heat transfer equation of fractal materials

A basic characteristic of fractal materials is the non-integer dimensions such as mass or "particle" dimensions [2]. For fractal materials the number of particles $N_D(W)$ or mass $M_D(W)$ in any region $W \subset \mathbb{R}^3$ of this material increase more slowly than the 3-dimensional volume $V_3(W)$ of this region. For the ball region W with radius R in an isotropic fractal material, this property can be described by the relation between the number of particles $N_D(W)$ in the region W of fractal material, and the radius R in the form $N_D(W) = N_0(R/R_0)^D$, where R_0 is the characteristic size of fractal material such as a minimal scale of self-similarity, and D is the "particle" dimension. If the fractal material consists of particles with identical masses m_0, then the relation $N_D(W) = N_0(R/R_0)^D$ can be represented in the form $M_D(W) = M_0(R/R_0)^D$, where $M_0 = m_0 N_0$. In this case, the mass dimension coincides with the "particle" dimension. The parameter D does not depend on the

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shape of the region W. Therefore fractal materials can be considered as a material with non-integer "particle" or mass dimension. An idealized model of fractal material is a medium distributed in empty space $\mathbb{R}^3$ with non-integer mass dimension $D < 3$. The fractal material can be considered as a fractal porous solid material.

For simplification, we consider scalar fields T and vector fields $\mathbf{q}$ that are independent of angles $T(\mathbf{r}) = T(r)$, $\mathbf{q}(\mathbf{r}) = \mathbf{q}(r) = q_r(r)\,\mathbf{e}_r$, where $r = |\mathbf{r}|$ is the radial distance, $\mathbf{e}_r = \mathbf{r}/r$ is the local orthogonal unit vector in the directions of increasing r, and $q_r = q_r(r)$ is the radial component of the vector $\mathbf{q}$. This means that we consider the case of only rotationally covariant functions, which is analogous to simplification, that is used for the NIDS integrations (see Section 4 of [21]).

Let us consider a region W_D in the fractal materials with the boundary $S_d = \partial W_D$ with dimensions $\dim(W_D) = D$ and $\dim(S_d) = d$. In general, the volume dimension D of the region and the boundary dimension d of this region are not related by the condition $d = D - 1$, $(\dim(\partial W_D) \neq \dim(S_d) - 1)$. It is convenient to define the parameter $\alpha_r = D - d$, which is a dimension of fractal material along the radial direction $\mathbf{e}_r$.

The gradient for the scalar field $T(\mathbf{r}) = T(r)$ and the radial dimension $\alpha_r \neq 1$ is defined [15] by the equation

$$\mathrm{Grad}_r^{D,d} T = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\, r^{\alpha_r - 1}} \frac{\partial T(r)}{\partial r} \mathbf{e}_r. \tag{1}$$

Applying integration in NIDS and the corresponding Gauss's theorem, the divergence is defined [15] in the form

$$\mathrm{Div}_r^{D,d} \mathbf{q} = \pi^{(1-\alpha_r)/2} \frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{\alpha_r - 1}} \frac{\partial q_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}} q_r(r) \right). \tag{2}$$

Using the operators (1) and (2) and $\mathbf{q} = q_r(r)\mathbf{e}_r$, we get [15] the scalar and vector Laplace operators by

$${}^S\Delta_r^{D,d} T = \mathrm{Div}_r^{D,d} \mathrm{Grad}_r^{D,d} T, \quad {}^V\Delta_r^{D,d} \mathbf{q} = \mathrm{Grad}_r^{D,d} \mathrm{Div}_r^{D,d} \mathbf{q}. \tag{3}$$

The scalar Laplacian for $d \neq D - 1$ for the field $T = T(r)$ is

$${}^S\Delta_r^{D,d} T = A(d, \alpha_r) \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 T}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r - 1}} \frac{\partial T}{\partial r} \right), \tag{4}$$

where

$$A(d, \alpha_r) = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r - 1/2}\,\Gamma((d+1)/2)}. \tag{5}$$

The vector Laplacian for $d \neq D - 1$ for the field $\mathbf{q} = v(r)\mathbf{e}_r$ is

$${}^V\Delta_r^{D,d} \mathbf{q} = A(d, \alpha_r) \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 q_r}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r - 1}} \frac{\partial q_r}{\partial r} - \frac{d\,\alpha_r}{r^{2\alpha_r}} q_r \right) \mathbf{e}_r. \tag{6}$$

The vector differential operators (1), (2), (4) and (6) allow us to describe fractal materials with the boundary dimensions $d \neq D - 1$ by continuous models with NIDS.

A heat transfer in fractal materials in the framework of continuous model with non-integer dimensional space is described by the equations

$$c_p\,\rho \frac{\partial T(\mathbf{r}, t)}{\partial t} = \lambda\, {}^S\Delta_r^{D,d} T(\mathbf{r}, t) + q_r(\mathbf{r}, t), \tag{7}$$

where $T = T(\mathbf{r}, t)$ is the local temperature (temperature field), λ is the thermal conductivity, c_p is the isobaric heat capacity (is specific heat capacity), ρ is the density of the material (ρc_p is considered as a volumetric heat capacity), ${}^S\Delta_r^{D,d}$ denotes the Laplace operator (4). Eq. (7) is the heat transfer equations for non-integer dimensional space and it can be used to describe the heat transfer in isotropic fractal materials.

The law of heat conduction, also known as the Fourier's law, describes the heat flux across a surface S_d. In the differential formulation of Fourier's law, the local heat flux is defined by the gradient for the scalar field $T(\mathbf{r}, t)$ and the radial dimension $0 < \alpha_r \leqslant 1$ by the equation

$$\mathbf{q}_v = -\lambda \mathrm{Grad}_r^{D,d} T(\mathbf{r}, t). \tag{8}$$

It is convenient to work in the dimensionless space variables $x/R_0 \to x, y/R_0 \to x, z/R_0 \to x, r/R_0 \to r$, that yields dimensionless integration and dimensionless differentiation in NIDS. Here R_0 is the characteristic size of a fractal material, such as a minimal scale of self-similarity of a considered fractal material. Then the density ρ and the fields $\mathbf{q}, p, f$ have correct physical dimensions.

The suggested generalizations of the heat transfer equation and Fourier's law describe thermal properties of fractal materials in the framework of continuous models with NIDS. These equations allow us to describe the heat transport in isotropic fractal materials, for the spherical or cylindrical symmetries, when the fields T and q_r are not depend on the angles.

3. Heat transfer in fractal pipe

Let us consider the heat transfer equation of fractal materials for a heat flow in fractal pipe of annular cross-section with the internal radius R_1 and external radius R_2. Using the continuous models with NIDS, we describe a steady heat flow in fractal pipe with circular cross-section. We assume that the axis of the pipe is the X-axis. The temperature field of fractal material is a function of r only. Using the fractal heat transfer Eq. (7), we have

$${}^S\Delta_r^{D,d} T(r) + \frac{q_r}{c_p\,\rho} = 0 \tag{9}$$

with $\alpha_r = D - d \neq 1$ in general.

Using (4), Eq. (9) is written in the form

$$A(d_x, \alpha_r) \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 T(r)}{\partial r^2} + \frac{d_x + 1 - \alpha_r}{r^{2\alpha_r - 1}} \frac{\partial T(r)}{\partial r} \right) + \frac{q_r}{c_p\,\rho} = 0, \tag{10}$$

where $A(d_x, \alpha_r)$ is defined by (5), $d_x = d - \alpha_x$, and α_x is dimension along the X-axis. Using $T(r)$ as a scalar field, we can apply equations, where $D \to D_x = D - \alpha_x$ and $d \to d_x = d - \alpha_x$ to get (10). Eq. (10) with $\alpha_r = \alpha_x = 1$ gives

$$\frac{\partial^2 T(r)}{\partial r^2} + \frac{d - 1}{r} \frac{\partial T}{\partial r} + \frac{q_r}{c_p\,\rho} = 0, \quad (1 < d < 2), \tag{11}$$

where $d = d_x + 1 > 1$ and $D = d + 1 > 2$.

The general solution of (10) is

$$T(r) = C_1 r^{\alpha_r - d_x} + C_2 - \frac{1}{2(d_x + \alpha_r)\alpha_r A(d_x, \alpha_r)} \frac{q_r}{c_p \rho} r^{2\alpha_r}$$
$$(1 < D < 3,\ \alpha_r - d_x \neq 0). \tag{12}$$

For non-fractal materials ($D = 3$), we have the solution in the form

$$T(r) = C_1 \ln(r) + C_2 - \frac{q_r}{4 c_p \rho} r^2. \tag{14}$$

The constants C_1 and C_2 in the general solution (12) are determined by the boundary conditions

$$T(R_1) = T_1 \quad T(R_2) = T_2. \tag{15}$$

These conditions give the equations

$$C_1 R_1^{\alpha_r - d_x} + C_2 - \frac{1}{2(d_x + \alpha_r)\alpha_r A(d_x, \alpha_r)} \frac{q_r}{c_p \rho} R_1^{2\alpha_r} = T_1, \tag{16}$$