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# An interface integral formulation of heat energy calculation of steady state heat conduction in heterogeneous media

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ABSTRACT

This paper presents a new boundary integral formulation of the heat energy calculation of steady state heat conduction in heterogeneous media using two methods, i.e. one which is based on some formulations from Hatta and Taya (1986) who investigated equivalent inclusion method, and another one which is closely similar to the method from Christensen (1979) who studied elastic inclusion problem. For the generalized self-consistent scheme (GSCS) model, the corresponding heat energy formulation is also given. Based on the obtained interface heat energy formulation, the boundary element method together with the Maxwell homogeneous scheme are used to calculate the effective heat conduction of 2D and 3D heterogeneous steady state heat conduction media.



## 1. Introduction

Steady state heat conduction in inhomogeneous medium has widely been studied using analytical methods or boundary element methods [4,8,15,5,16]. Further research about steady state heat conduction inhomogeneous problems is to carry out the analysis of shape optimization of inhomogeneities based on the minimization of objective function, i.e. heat energy of inhomogeneous medium. In published literature [8], the heat energy formulation from equivalent inclusion method contains domain integrals which cannot be analytically calculated for inhomogeneities of complex geometrical shapes. By means of numerical methods, i.e. finite element method [9], these domain integrals can be solved by discretizing the inhomogeneous domain into many finite elements. Following the method from Christensen [6] for elasticity, the heat energy calculation formulation of steady state heat conduction inhomogeneous problems can be obtained, in which only the interface integrals appeared. This formulation is applicable to the inhomogeneities with various geometries, but the temperature and heat flux on the inhomogeneities-matrix interfaces have to be computed in advance using numerical methods, e.g. finite element method [9] or boundary element method [19].

This paper develops a simpler heat energy calculation formulation for steady state heat conduction inhomogeneous problems using two existing methods, i.e. Hatta and Taya's method (1986) and Christensen's method (1979). The present formulation only contains the inhomogeneities-matrix interface integrals and only the temperature on the interfaces is needed in calculation of the heat energy of the studied problems. This formulation is especially suitable to investigate the homogenization and shape optimization of steady state heat conduction inhomogeneous problems. Application of the present heat energy increment integral formulation together with the boundary element method and the Maxwell homogenization scheme is shown in numerical examples.

## 2. Basic formulations

### *2.1. The heat energy interface integral formulation based on Hatta and Taya's method (1986)*

Following the work carried out by Hatta and Taya [8], the heat energy formulation of steady state heat conduction in inhomogeneous problems can be written as follows

$$W = \frac{1}{2}\int_D q_i^0 T_{,i}^0 dv + \frac{1}{2}\int_\Omega q_i^0 T_{,i}^* dv \qquad (1.1)$$

where $D$ is the entire composite body, $\Omega$ is the inclusion domain, $T_{,i}^*$ is the eigentemperature gradient, $T^0$ is the uniform temperature field, $q_i^0$ is a constant heat flux. The usual Einstein's summation convention is used throughout this paper.

The second integral on the right hand side of Eq. (1.1) denotes the heat energy increment $\Delta W$ due to the existence of the inclusions, i.e.

$$\Delta W = \frac{1}{2}\int_\Omega q_i^0 T_{,i}^* dv \qquad (1.2)$$

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Eq. (20) appearing in Hatta and Taya [8] is shown as

$$k_M\delta_{ij}(T^0_{,j}+\tilde{T}_{,j}+T^c_{,j}-T^*_{,j})=K^I_{ij}(T^0_{,j}+\tilde{T}_{,j}+T^c_{,j})\quad \text{in } \Omega \tag{1.3}$$

where $K^I_{ij}=k_I\delta_{ij}$ for homogeneous inclusion in which $\delta_{ij}$ is the Kronecker's delta. $k_I$ and $k_M$ are the heat conductivities of the inclusion and matrix, respectively. The actual temperature $T$ is equal to $T^0+\tilde{T}+T^c$ in which $T^c$ and $\tilde{T}$ are the disturbance of temperature $T^0$ due to the $I$th inclusion and its eigentemperature, respectively.

Considering $k_M\delta_{ij}(T_{,j}-T^*_{,j})=k_I\delta_{ij}T_{,j}$, Eq. (1.3) can be rewritten as

$$k_M(T_{,j}-T^*_{,j})=k_IT_{,j} \tag{1.4}$$

Therefore, we can obtain the following equation from Eq. (1.4)

$$T^*_{,j}=\frac{k_M-k_I}{k_M}T_{,j} \tag{1.5}$$

Eq. (1.5) is substituted into Eq. (1.2), one has

$$\Delta W=\frac{1}{2}\sum_{I=1}^{N}\frac{k_M-k_I}{k_M}\int_{\Omega_I}q^0_iT_{,j}dv$$
$$=\frac{1}{2}\sum_{I=1}^{N}\frac{k_M-k_I}{k_M}\int_{\Gamma_I}q^0_in_iTd\Gamma \tag{1.6}$$

where $\Gamma$ is the matrix-inclusion interface, $n_i$ is the component of a unit outer normal vector to the surface of the domain $\Omega$. The symbol $I$ denotes the $I$th inclusion. $N$ is the inclusion number.

From Eq. (1.6), one can observe that if $k_I=0$ and $N=1$, then Eq. (1.6) becomes $\Delta W=\frac{1}{2}\int_\Gamma q^0_in_iTd\Gamma$ which is the same as the one derived by Rodin [17]. If $k_I=k_M$, then $\Delta W=0$ as expected.

Based on Eq. (1.6), the inclusion shape of minimizing the heat energy can be done for 2D and 3D inhomogeneous problems (infinite domain and finite domain).

### *2.2. The heat energy interface integral formulation based on Christensen's method (1979)*

For clarify, the derivation of the heat energy interface integral formulation of steady state heat conduction of inhomogeneous problems closely follows the method of Christensen [6] for elastic heterogeneity.

Fig. 1 shows two matrices, i.e. one with the inclusion, and one without the inclusion. $q_i(i=x,y,z)$ is the heat flux along the $i$th direction.

The heat energy in the heterogeneous medium (Fig. 1(a)) is

$$W=\frac{1}{2}\int_D q_iT_{,i}dv \tag{2.1}$$

where the adopted symbols in Eq. (2.1) and the following equations are the same meaning as those in Hatta and Taya's method (1986)

The heat energy in the homogeneous medium (Fig. 1(b)) is

$$W_0=\frac{1}{2}\int_D q^0_iT^0_{,i}dv \tag{2.2}$$

Using Eq. (2.2), Eq. (2.1) can be rewritten as

$$W=W_0+\frac{1}{2}\int_D(q_iT_{,i}-q^0_iT^0_{,i})dv \tag{2.3}$$

Based on the divergence theorem and the equilibrium equations $q_{i,i}=0$ and $q^0_{i,i}=0$, Eq. (2.3) can further be rewritten as

$$W=W_0+\frac{1}{2}\int_\Gamma(n_iq_iT-n_iq^0_iT^0)d\Gamma \tag{2.4}$$

where $\Gamma$ is the surface of the whole domain $D$.

On the surface $\Gamma$,

$$n_iq_i=n_iq^0_i\quad \text{on } \Gamma \tag{2.5}$$

Eq. (2.4) becomes using Eq. (2.5)

$$W=W_0+\frac{1}{2}\int_\Gamma n_iq^0_i(T-T^0)d\Gamma \tag{2.6}$$

The inclusion as shown in Fig. 1(a) can be replaced by one same domain with the matrix material having a distribution of heat fluxes as shown in Fig. 2.

Fig. 2 can be decomposed into two parts as shown in Fig. 3(a) and (b).

The field variables in Fig. 2 are $\bar{q}_i$, $\bar{T}_{,i}$ and $\bar{T}$, whereas those in Fig. 3(b) are $q'_i$, $T'_{,i}$ and $T'$ .

Considering the problem in Fig. 3(a) being the same as that in Fig. 1(b), the following relationships can be obtained

$$\begin{aligned}\bar{q}_i&=q^0_i+q'_i\\ \bar{T}_{,i}&=T^0_{,i}+T'_{,i}\\ \bar{T}&=T^0+T'\end{aligned} \tag{2.7}$$

The heat energy in Fig. 2 is

$$W=\frac{1}{2}\int_D(q^0_i+q'_i)(T^0_{,i}+T'_{,i})dv \tag{2.8}$$

Eq. (2.8) can be rewritten as

$$W=W_0+W'+W_{INT} \tag{2.9}$$

where

$$W'=\frac{1}{2}\int_D q'_iT'_{,i}dv \tag{2.10}$$

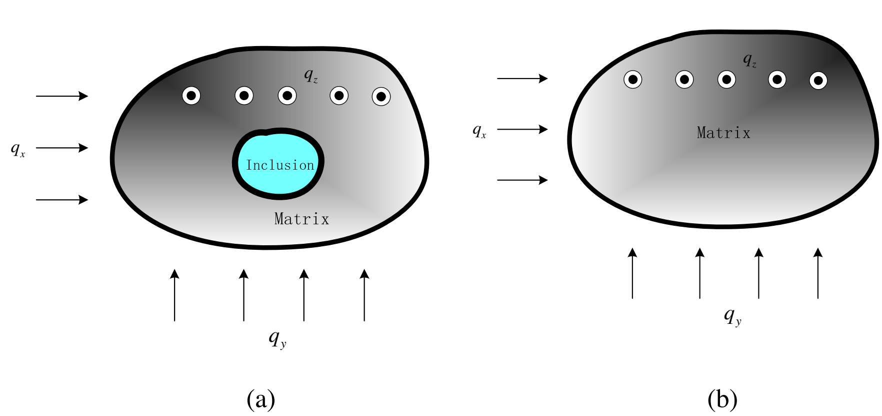


**Fig. 1.** (a) Matrix with inclusion; (b) matrix without inclusion.