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An analytic expression for radiation view factor between two arbitrarily oriented planar polygons

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ABSTRACT

An analytic expression is derived for the radiation view factor between two arbitrarily oriented planar triangles and, by a simple generalization, planar polygons. Unlike most attempts so far, which use the contour integration technique, Nusselt's unit sphere method is used in this work. Two important features of the analytical expression derived here are that: (1) the use of numerical quadrature is not necessary for the computation, and (2) the symmetry of the expression ensures that reciprocity of view factors is preserved.

1. Introduction

Radiation view factors, also known as shape factors or configuration factors or form factors, play a key role in the computation of radiation heat transfer [1,2] as well as global illumination in computer graphics [3]. The view factor between two surfaces is defined as the fraction of diffuse radiation from one surface that is intercepted by the second surface. Mathematically, it is given by:

$$F_{12} = \frac{1}{A_1}\int_{A_1} dA_1 \frac{1}{\pi}\left[\int_{A_2} \cos\theta_1 \frac{\cos\theta_2 dA_2}{R^2}\right] \quad (1)$$

Unless the configuration of surfaces is such that analytic expressions are available in literature [1], view factors have to be computed numerically. The most general method for computing the four dimensional integral in Eq. (1) is the Monte Carlo method [4]. Since Monte Carlo simulations can be too expensive for large three-dimensional configuration of surfaces, efficient deterministic methods to evaluate Eq. (1) have been developed. The four integrations necessary to compute Eq. (1) can be reduced to two integrations by using Stokes theorem of vector calculus [5,6,1,2] or, in certain cases, by using Nusselt's unit sphere method [7,8,1,2]. On using Stokes theorem, the resultant expression for F_{12} is a double contour integral along the edges of the surfaces between which F_{12} is to be determined. When the contour along one of the surfaces can be decomposed into straight lines, the integral along that contour can be calculated analytically by using the technique described by Mitalas and Stephenson [9]. Nusselt's unit sphere method, also known as the projection technique, results in a surface integral on surface 1. Many analytic and numerical results for diffuse view factors between specific configurations of planar surfaces (and non–planar too) are given in Ref. [10], and references therein. Mathiak [11], Ambirajan and Venkatesh [12], Mazumder and Ravishankar [4] use the contour integral method and numerical quadrature to obtain view factors between arbitrarily oriented planar polygons.

The goal for this work was to determine if an analytic expression exists and, if so, to calculate the view factor between two arbitrarily oriented planar triangles or polygons. The motivation for considering these shapes comes from mesh generation software used for finite element analysis and computer graphics where most arbitrary surfaces are discretized into planar triangular or polygonal facets. In this work, two triangles are used to aid the derivation of an analytic expression for view factor; the extension to planar polygonal shapes requires a trivial change (see last sentence of Section 3.2). The configuration of the two arbitrarily oriented planar triangles, entirely visible to each other, is shown in Fig. 1. Without loss of generality, triangle 1 is assumed to lie in the $z = 0$ plane. The vertices of triangle 1 have position vectors $\vec{r}_1^{(1)}, \vec{r}_2^{(1)}, \vec{r}_3^{(1)}$, and those on triangle 2 have position vectors $\vec{r}_1^{(2)}, \vec{r}_2^{(2)}, \vec{r}_3^{(2)}$. As Lipps [13] has suggested, the view factor between the two planar triangles should be a function of the coordinates of the vertices of both triangles. In fact, Schroder and Hanrahan

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Nomenclature

$A_P(\vec{\boldsymbol{x}})$	$\frac{1}{\pi}\mid A_P(\vec{\boldsymbol{x}})\mid$ is the view factor of triangle 2 at $\vec{\boldsymbol{x}}$.
$A_P^{(pq)}(\vec{\boldsymbol{x}})$	contribution to $\mid A_P(\vec{\boldsymbol{x}})\mid$ from line between $\vec{\boldsymbol{r}}_p^{(2)}$ and $\vec{\boldsymbol{r}}_q^{(2)}, p=1,2,3; q=P(p)$.
A_1	area of surface 1
$C(A,B,C,A)$	a closed contour through the points A,B,C,A in that order
$Cl_2^{(N)}(\theta)$	approximation to Clausen's integral with Chebyshev polynomials up to order $2N+1$
$Cl_2(\theta)$	Clausen's integral. See Eq. (26) for definition
F_{12}	view factor of surface 2 at surface 1
$G_{ij,k}^{(pq)}$	see Eq. (15) for definition $(k=1,2)$
L	non-dimensionalized version of l, see Eq. (14)
$Li_2(z)$	dilogarithm function, defined as $-\int_0^1 dt\,\frac{\log(1-zt)}{t}$
$P(i)$	cyclical permutation of i. $P(i)=2,3,1$ when $i=1,2,3$.
R	distance between two differential area elements on A_1 and A_2
S	non-dimensionalized version of s, see Eq. (14)
$T_n(\theta)$	Chebyshev polynomial of order n. $T_n(\theta)=\cos n\theta$
P	$L+\sqrt{L^2+1}$
W	$S+\sqrt{S^2+1}$
b_n	coefficient of T_{2n+1} in expansion of Cl_2
$d_{ij,pq}$	$\mid d_{ij,pq}\mid$ is the distance between $\vec{\boldsymbol{r}}_{ij,pq}^{(\perp,1)}$ and $\vec{\boldsymbol{r}}_{pq,ij}^{(\perp,2)}$
dA_i	differential area element on surface i $(i=1,2)$
$\tilde{f}(S,L)$	see Eq. (22b) for definition
$f(S,L)$	see Eq. (21b) for definition
$g_{ij,k}^{(pq)}$	see Eq. (14) for definition $(k=1,2)$
$g_{pq}(x,y)$	see Eq. (8) for definition
$\vec{\boldsymbol{l}}_{pq}$	unit vector from $\vec{\boldsymbol{r}}_p^{(2)}$ to $\vec{\boldsymbol{r}}_q^{(2)}$
$\hat{\boldsymbol{l}}_{pq,ij}$	unit vector along the line joining $\vec{\boldsymbol{r}}_p^{(2)}$ and $\vec{\boldsymbol{r}}_q^{(2)}$
l,l'	parametrization variables for each edge along triangle 2
$\vec{\boldsymbol{r}}_i^{(n)}$	position vector of vertex i $(i=1,2,3)$ on triangle n $(n=1,2)$.
$\vec{\boldsymbol{r}}_{ij,pq}^{(\perp,1)}$	point on line joining $\vec{\boldsymbol{r}}_i^{(1)}$ and $\vec{\boldsymbol{r}}_j^{(1)}$ which is closest to line joining $\vec{\boldsymbol{r}}_p^{(2)}$ and $\vec{\boldsymbol{r}}_q^{(2)}$
$\vec{\boldsymbol{r}}_{pq,ij}^{(\perp,2)}$	point on line joining $\vec{\boldsymbol{r}}_p^{(2)}$ and $\vec{\boldsymbol{r}}_q^{(2)}$ which is closest to line joining $\vec{\boldsymbol{r}}_i^{(1)}$ and $\vec{\boldsymbol{r}}_j^{(1)}$
$\vec{\boldsymbol{s}}_{ij}'$	unit vector from $\vec{\boldsymbol{r}}_i^{(1)}$ to $\vec{\boldsymbol{r}}_j^{(1)}$
$\hat{\boldsymbol{s}}_{ij,pq}$	unit vector along the line joining $\vec{\boldsymbol{r}}_i^{(1)}$ and $\vec{\boldsymbol{r}}_j^{(1)}$
s,s'	parametrization variables for each edge along triangle 1
$(x_p^2, y', z_p^{(2)})$	coordinate of point $\vec{\boldsymbol{r}}_p^{(2)}, p=1,2,3$
$(x_q^2, y', z_q^{(2)})$	coordinate of point $\vec{\boldsymbol{r}}_q^{(2)}, q=1,2,3$
$\vec{\boldsymbol{x}}$	position vector of location on triangle 1 $(=x\hat{x}+y\hat{y})$
$x_<^{(pq)}$	smallest value of x when x-axis is parallel to projection of $\vec{\boldsymbol{r}}_q-\vec{\boldsymbol{r}}_p$ on $z=0$ plane
$x_>^{(pq)}$	largest value of x when x-axis is parallel to projection of $\vec{\boldsymbol{r}}_q-\vec{\boldsymbol{r}}_p$ on $z=0$ plane
$y_<(x)$	lower limit of integration along y at constant x
$y_>(x)$	upper limit of integration along y at constant x
$U_p^{(2)}$	projection of $\vec{\boldsymbol{r}}_p^{(2)}$ $(r=1,2,3)$ on a unit hemisphere centered at $(x,y,0)$ on triangle 1
α	$\cos^{-1}(\hat{\boldsymbol{s}}_{ij,pq}\cdot\hat{\boldsymbol{l}}_{pq,ij})$
$\mathfrak{A}_P(x)$	$\frac{1}{2}\mid(x-x_p^{(2)})z_q^{(2)}-(x-x_q^{(2)})z_p^{(2)}\mid$
θ_i	angle between the surface normal to triangle i and the position vector from a differential element on triangle i to one on the other triangle.
$\vec{\boldsymbol{\rho}}_i^{(2)}$	$\vec{\boldsymbol{r}}_i^{(2)}-\vec{\boldsymbol{x}}(i=1,2,3)$

[3] had affirmed that it was indeed possible to find analytic expressions for view factors between planar polygons.

The result obtained here is similar to that of Schroder and Hanrahan, though the path taken is different. First, Schroder and Hanrahan use the contour integral method as the starting point. The unit sphere method is used here. The primary reason for this choice over the contour integral method was our appreciation for the geometric appeal and interpretation of the unit sphere method. Second, the expression derived by Schroder and Hanrahan appears complicated because of the presence of complex square roots [14], the signs of which have to be chosen carefully. The expression for view factors derived in this work is, in the author's assessment, in a form more suitable for use in numerical codes. The expression for view factor is rendered in a symmetric form so that it becomes apparent that the reciprocity of view factors between two surfaces is obeyed.

As derived by Schroder and Hanrahan, the final expression for view factor between two planar polygons contains the dilogarithm function [15], the evaluation of which necessitates the usage of numerical quadrature [4]. In this work, it is shown that what is needed is not the dilogarithm function itself, but the imaginary part of the dilogarithm function. As shown by Lewin [16], the imaginary part of the dilogarithm function is simpler to evaluate than the real part. Since the imaginary part of the dilogarithm function can be computed without any numerical integration (see Section 4), it qualifies as an analytic result.

The structure of the rest of this paper is as follows: In Section 2, an analytic expression for the view factor between an infinitesimal planar area and a planar triangle is determined using Nusselt's unit sphere method. In Section 3, the analytical expression for view factor between two planar triangles is derived by integrating (twice) the result from Section 2. The first of the two integrations (see Section 3.1) is relative simple and results in an expression for the view factor in the form of a line integral along the edges of one of the triangles. A parametrization of the edges of the triangles is introduced in Section 3.2, using which an analytical expression for the view factor between two planar triangles is derived. But for the dilogarithm function, the other terms of the expression are elementary functions. In Section 4, the dilogarithm function is replaced by Clausen's integral, which can be evaluated easily

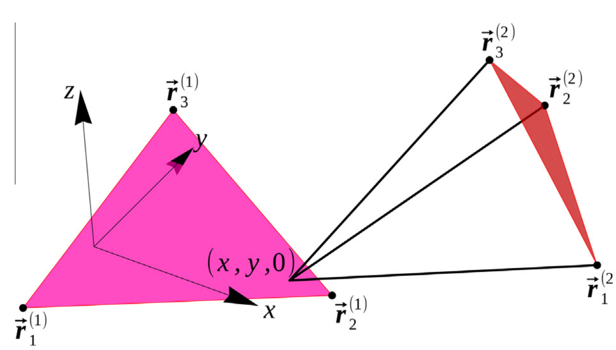

Fig. 1. Configuration of two arbitrarily oriented planar triangles. One of the triangles (triangle 1) is assumed to be in the $x-y$ plane. The vertices of the two triangles can be represented as $\vec{\boldsymbol{r}}_i^{(j)}, i=1,2,3$ and $j=1,2$. The x and y axes can be oriented arbitrarily in the plane of triangle 1.