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Constriction resistance of a plated solid

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ABSTRACT

Conduction of heat between two solids is frequently restricted by surface roughness to flow through microscopic areas of contact. The contact areas can be of any shape, but some idea of their effect can be gained by idealising them as isolated circular areas. Then for conduction between two large, uniform bodies through a contact of radius a there is a 'constriction resistance' due to the concentration of the flow lines equal to $1/2aK$ where K is the thermal conductivity. This paper investigates how the constriction resistance is modified by a plated coating of different thermal conductivity. The method of finding an upper bound to the resistance by using an arbitrary distribution of heat flux across the contact and calculating and minimising the integral $H = \int_V \{|\mathbf{q}|^2/K\}dV$ (well-known for use in electrical flow problems) is established, and it is shown that this is a highly efficient method of solving the problem, producing answers which while being upper bounds are also highly accurate values, and the method can be strongly recommended for use in other problems. The results for all values of the ratio of spot size to plating thickness a/d and of the thermal conductivity ratio $\kappa \equiv K_a/K_b$ are presented in a plot of "universal plating factors".

The idea that the heat flux through the contact is simply a combination of the distribution without the coating $\left(q \propto 1/\sqrt{a^2 - r^2}\right)$ and a uniform flux, although it leads to acceptable values for the constriction resistance, seems not to represent the real physical picture.

1. Introduction

The flow of heat, or electric current, between two solids is frequently restricted to the small microscopic contact areas caused by surface roughness; and while these can be of any shape, some idea of their effect may be gained by treating them as isolated circular contacts on the surface of planar half-spaces. For two half-spaces of the same, uniform, material the effect is well-known: there is a 'constriction resistance' due to the concentration of the flow lines equal to $1/2aK$ where K is the thermal conductivity and a the contact radius (or $\rho/2a$, where $\rho \equiv 1/\sigma$ is the electrical resistivity). Even when the two bodies have different, but still uniform, properties, the same distribution of flow over the contact (as $q = Q/(2\pi a\sqrt{a^2 - r^2})$), taken as input to each body separately, gives a uniform temperature over the contact, so that this is the exact solution for the pair, and the combined constriction resistance is *exactly* $1/4aK_1 + 1/4aK_2$. [From this point, the terminology will be thermal: but the mathematical potential problems are identical, and for the electrical problem it is merely necessary to translate 'heat flow' as 'current', 'temperature' as 'potential', and replace $1/K$ by ρ.]

In practice the two bodies in contact are often not uniform, but are protected by a surface layer of different conductivity. (We ignore the very real possibility that the plated layer may not have uniform conductivity). Here, the resistance of a single such plated conductor is examined (Fig. 1).

We give an approximate solution, based on the principle that if the integral

$$H = \int_V \{|\mathbf{q}|^2/K\}dV$$

where $\mathbf{q} = K grad\theta$ is calculated for an arbitrary distribution of heat flux over the contact area, this will provide an upper bound to Q^2R (see Appendix A for proof).

This is well-known in electrical theory, where it is simply a statement of the principle that the true current distribution minimises the Joule heat production $\int_V \rho|\mathbf{J}|^2dV$ in the body. Thus, instead of solving the exact problem where the contact is an isothermal, we find an *exact* solution but for an *arbitrary* distribution of heat flux over the contact, and so obtain an upper bound to the real problem. Of course we then vary the arbitrary heat flux, and take the minimum value of H/Q^2 as the resistance. Experience suggests that besides being an upper bound, this is usually a rather good estimate of the resistance.

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Nomenclature

a	contact radius
d	plating thickness
$f(\lambda)$	Hankel transform of flux $q(r)$
$j_n(t)$	$\sqrt{\pi/2}\, J_{n+1/2}(t)/t^{1/2}$ (spherical Bessel function)
m	$=(\kappa-1)/(\kappa+1)$ or dummy elliptic integral parameter
q	heat flux over contact
q_p	$=(1-r^2/a^2)^p$
(r, z)	co-ordinate system
$C_\nu(t)$	$=2^\nu \nu!\, J_\nu(t)/t^\nu$ (reduced Bessel function)
$B(m)$	complete elliptic integral $mB(m) \equiv E(m)-(1-m)K(m)$
$E(m)$	complete elliptic integral
$G(\lambda d)$	$=\frac{\exp(2\lambda d)+m}{\exp(2\lambda d)-m}$
H	$\int(\lvert q\rvert^2/K)\, dV$ Quantity to be minimised. Corresponds to Joule heat $\int(\rho \lvert J\rvert^2)\, dV$
$J_\nu(t)$	Bessel function of order ν
$K(m)$	complete elliptic integral
K	thermal conductivity
K_a	thermal conductivity of plate
K_b	thermal conductivity of substrate
Q	total heat flow through contact
$R_{\alpha\beta}$	Resistance coefficient with input q_α and temperature θ_β.
R_{pn}	resistance coefficient $\int_0^a q_p(r)\,\theta_n(r)\,2\pi r\, dr$
R^0_{pn}	resistance coefficient if conductivity uniform.
ΔR_{pn}	addition due to conductivity discontinuity
R_{sub}	resistance for a body of substrate conductivity
R_{pl}	resistance for a body of plate conductivity
U	universal plating factor$(R-R_{sub})/(R_{pl}-R_{sub})$
$\alpha,\ \beta$	generic subscripts: later, Boussinesq and uniform heat flux
κ	$=K_a/K_b$ (plate conductivity)/(substrate conductivity)
θ	temperature
θ_n	temperature resulting from heat flux q_n
$\varsigma(n)$	Riemann zeta function (NBS Table 23.3)

Specifically, to find the resistance for the plated body, we use a linear combination of arbitrary distributions of heat flux over the circular contact area, but then optimise the coefficients to minimise the volume integral. Fortunately it is not necessary to actually find the temperature or heat flux throughout the body: the volume integral can be converted into a surface integral by Gauss' theorem:

$$\int \mathbf{q}(\mathbf{r})|^2/K\, dV \equiv \int \mathbf{q}(\mathbf{r}).grad\theta dV = \int div(\theta\mathbf{q}(\mathbf{r}))dV = \int \theta\mathbf{q}(\mathbf{r}).dS.$$

Thus only the *surface* heat flux (chosen) and surface temperatures (calculated) are needed. For a heat flux entering over a given area ("the contact") and leaving at zero temperature at "infinity", all that need be found is the temperature distribution over the contact. Then for a linear combination of heat fluxes $q(r) = aq_1(r) + bq_2(r) + cq_3(r) + \ldots$ giving rise to a temperature over the contact of $\theta(r) = a\theta_1(r) + b\theta_2(r) + c\theta_3(r) + \ldots$, the integral H will equal

$$\int (aq_1+bq_2+cq_3+\ldots)(a\theta_1+b\theta_2+c\theta_3+\ldots)dS = a^2R_{11}+2abR_{12}+\ldots$$

where $R_{ij} = \int q_i\theta_j dS$ and we have used the fact that $\int q_i\theta_j dS = \int q_j\theta_i dS$. [We shall refer to such quantities as "Resistance coefficients.] The quadratic form is minimised, subject to the condition that the total heat flux is fixed: for this it is convenient to use Lagrange's method of undetermined multipliers.

For the simple case of the combination of two arbitrary input heat flux distributions (q_α, q_β) the minimisation is elementary, and leads to the simple equation [6]:

$$R_{\min} = \frac{(R_{\alpha\alpha}R_{\beta\beta} - R^2_{\alpha\beta})}{R_{\alpha\alpha}+R_{\beta\beta}-2R_{\alpha\beta}} \equiv R_{\alpha\alpha} - \frac{(R_{\alpha\alpha}-R_{\alpha\beta})^2}{R_{\alpha\alpha}+R_{\beta\beta}-2R_{\alpha\beta}}.$$

It should again be emphasised that we never actually find the distribution of heat flux or temperature *throughout* the body: only the values over the circular contact area are needed to obtain $\int_{r=0}^{a} 2\pi\theta q r dr$. The "averaging" of the temperature over the contact $\int_{r=0}^{a}\theta(r)[rq]dr$ may also be done analytically, greatly reducing the amount of numerical integration necessary since the individual temperatures are never needed.

An alternative method of estimating the resistance is to find the *mean* temperature over the contact due to an imposed heat input distribution, as noted by Carslaw and Jaeger [2]. This will certainly give a reasonable estimate: but how good is not clear. Negus et al. [8] improve on this in a solution of the present problem. They determine the temperature distributions due to two input fluxes $q \sim (1-r^2/a^2)^{-1/2}$ and q uniform, (which we shall denote as q_α and q_β) and find the combination with the minimum variation over the contact: the corresponding combination of the two mean temperatures is then taken as the resistance estimate. Initially these same two heat flux distributions were used here, and then since q_β is uniform, the two means used by Negus et al. will be our $R_{\alpha\beta}$ and $R_{\beta\beta}$. It is satisfactory to report almost perfect agreement between our values of these and their values, despite the different procedures used for finding them: Negus et al. do not find the equivalent of our $R_{\alpha\alpha}$ (a *weighted* mean). But because it proves to be easier to calculate the mean (even weighted) than even a single individual value, the present method requires considerably less calculation: and has the merit of producing a definite upper bound. The direct use of a mean temperature *may* well give a better estimate; but all that is then certain is that both the exact answer and, of course, the mean, lie in the range between the least and greatest values of the temperature.

2. General solution

We consider a body $z \geqslant 0, 0 \leqslant r < \infty$ with a plating of thickness d of conductivity K_a, on a substrate with conductivity K_b. The

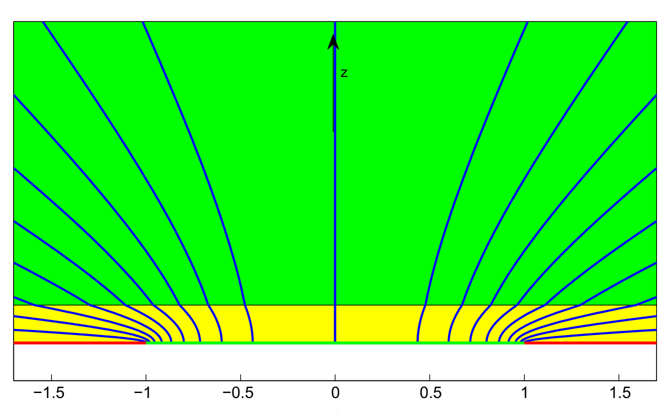

Fig. 1. Flux lines as heat enters a half-space through a circular contact area. [The flux lines will be kinked where the conductivity changes between plating and substrate].