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On-line detecting heat source of a nonlinear heat conduction equation by a differential algebraic equation method

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ABSTRACT

We consider an inverse problem for detecting an unknown time-dependent heat source in real-time for a nonlinear heat conduction equation, with the aid of an extra measurement of temperature at an internal point. After a finite difference discretization of governing equation into ordinary differential equations, we recast them and the measured data as a set of differential algebraic equations (DAEs), which is a novel view of the inverse heat source problem. Then we solve the resultant DAEs by a $GL(n,\mathbb{R})$ Lie-group method, which can be used as an on-line estimator to detect unknown heat source of nonlinear heat conduction equation, by using only a real-time measurement of internal temperature under a randomly noisy disturbance. The estimated results obtained by the novel Lie-group differential algebraic equations (LGDAE) method are quite promising and robust enough against large random noises.

1. Introduction

The inverse problems nowadays range over many scientific fields, which include solid mechanics, fluid dynamics and heat transfer, to name a few. The inverse heat conduction problems (IHCPs) are popularly used in many practical industrial heat conduction engineering. To be used efficiently, the IHCP has to take nonlinear governing equations into account and to give a real-time estimation of the unknowns.

The present study aims to estimate as accurately as possible the unknown time-varying heat source in a manner of real-time detection by solving an inverse heat conduction problem under an overspecified internal data. The estimation is based on a transient temperature measurement undertaken by a thermocouple mounted on an internal point of a heat conducting rod, which is avoidably being polluted by a random noise.

Applications of inverse methods span over many related heat transfer topics. Sometimes the temperature and heat flux data on the boundary are known and one wants to determine the material properties. Those problems are often referred to as parameter identification problems in the literature [1,2]. Most inverse problems belong to a family of problems that have an inherent ill-posed property.

The parameter determination in partial differential equations from overspecified data plays a crucial role in applied mathematics and physics. Those problems are widely encountered in the modelling of physical phenomena [3–6]. Here, we consider an inverse heat source problem to find an unknown heat source $H(t)$ in a one-dimensional nonlinear heat conduction equation, of which one needs to find the temperature distribution $T(x,t)$ as well as the heat source $H(t)$ that simultaneously satisfy

$$T_t(x,t) = G(T_{xx}(x,t), T_x(x,t), T(x,t)) + H(t),$$
$$0 < x < \ell, \quad 0 < t \leqslant t_f, \tag{1}$$
$$T(0,t) = F_0(t), \quad T(\ell,t) = F_\ell(t), \tag{2}$$
$$T(x,0) = f(x). \tag{3}$$

Because the problem has an unknown function $H(t)$, it cannot be solved directly. In the above, ℓ is a length of the heat conducting rod, and t_f is a terminal time. $G(T_{xx}(x,t), T_x(x,t), T(x,t))$ is a linear function of T_{xx} but can be a nonlinear function of $T_x(x,t)$ and $T(x,t)$.

A new method will be developed to estimate the unknown heat source $H(t)$ in real-time, which is subjected to the above boundary conditions and initial condition, as well as an overspecified temperature measurement at an internal point x_m:

$$T(x_m,t) = F_m(t), \tag{4}$$

where $0 < x_m < \ell$.

For the inverse problem governed by linear heat conduction equations there are many studies as can be seen from the papers by Cannon and Duchateau [7] for identifying $H(u)$, and Savateev [8] and Borukhov and Vabishchevich [9] for identifying $H(x,t)$ with additive or seperable space and time. Many researchers sought the

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Nomenclature

$\mathbf{A}$	coefficient matrix defined in Eq. (12)
$\mathbf{a}, \mathbf{b}$	vectors defined in Eq. (13)
c	$:= \mathbf{a} \cdot \mathbf{b}$
$\mathbf{f}$	n-dimensional vector field
$\bar{\mathbf{f}}$	$:= \mathbf{f}(\bar{x}, \mathbf{y}, \bar{t})$
$f(x)$	initial temperature function
$F_0(t)$	left-boundary temperature function
$F_\ell(t)$	right-boundary temperature function
$F_m(t)$	temperature function at x_m
$\mathbf{F}$	constraints equation
$GL(n, \mathbb{R})$	n-dimensional linear group
$\mathbf{G}$	an element of $GL(n, \mathbb{R})$
h	time stepsize
$H(t)$	time-dependent heat source
H_k	$:= H(t_k)$
$\mathbf{I}_n$	n-dimensional unit matrix
ℓ	length of rod
$\Vert \bullet \Vert$	Euclidean norm
n	number of discretized spatial points
$R(i)$	random numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
s	level of noise
t	time
t_f	final time
$\bar{t}$	$:= t_0 + \theta h$
T	temperature
$\mathbf{T}$	temperature vector of T^i
$\bar{\mathbf{T}}_k$	$:= (1-\theta)\mathbf{T}_k + \theta \mathbf{T}_{k+1}$
$T_i(t)$	$:= T(x_i, t)$
w	$:= \mathbf{b} \cdot \mathbf{x}$
x	space variable
Δx	mesh size of x
x_m	temperature measuring point
x_i	$:= i\Delta x$
$\bar{\mathbf{x}}_k$	$:= (1-\theta)\mathbf{x}_k + \theta \mathbf{x}_{k+1}$
$\mathbf{y}_k$	vector of parameter
$\mathbf{z}_{k+1}$	a tentative value of $\mathbf{x}_{k+1}$
Greek symbols	
$\varepsilon_1, \varepsilon_2$	convergence criteria
η	coefficient defined in Eq. (22)
θ	weighting factor
Subscripts and superscripts	
i	index
j	index
k	index
m	index
n	index
q	index
T	transpose

heat source as a function of only space or time, for example, Farcas and Lesnic [10], Ling et al. [11], Yan et al. [12], Yang and Fu [13], Yang et al. [14], and Kuo et al. [15].

The model problem presented here used to describe a heat transfer process with a time-dependent heat source produces the temperature at a given point x_m in the spatial domain at time t. Thus, the purpose of solving this inverse problem can be viewed as an inverse real-time control problem to identify the source control function on-line that produces at the current time a desired temperature at a given point x_m in the spatial domain. The traditional approach in solving the problem of this sort approximately consists in a reduction of it to the first kind Volterra integral equation, and then some regularization techniques are used to solve this ill-posed problem. According to this type formulation, Maalek Ghaini [16] has proven the existence, uniqueness and stability problems; however, no numerical procedures and examples were presented. More interestingly, Yan et al. [12] have transformed the linear problem into a three-point boundary value problem.

The heat source identification of $H(t)$ in real-time is one of the inverse problems for the applications in heat conduction engineering by detecting the thermal source on-line. For the present inverse problem of a real-time heat source identification the observed effect is the temperature measurement $T(x_m, t)$ at an internal point $x = x_m$ in the rod. When we can measure the internal temperature in real-time, we can identify the heat source immediately in the time domain. We are interesting to search the cause of the unknown heat source $H(t)$ in Eq. (1), which induces the effect we observe through measurement. For the inverse heat source problems the measurement error may often lead to a large discrepancy from the true cause. On the one hand, the un-regularized method is fast but inherently unstable [17]. On the other hand, methods which provide acceptable results require a quite complicated procedure of regularization and more future information, which are not available in a real-time control environment. In this paper we are going to develop a real-time identification method of the unknown heat source on-line, which is a more difficult inverse problem than that with an off-line identification of the unknown heat source. The algorithm in a real-time system must satisfy the following requirements: stability of the solution without using future data, computational efficiency, accuracy, and the representation of solution in the time domain.

Liu [18] has developed a two-stage Lie-group shooting method (TSLGSM) for three-point boundary value problems of second-order ordinary differential equations. Then, Liu [19] and Yeih and Liu [20] have used the TSLGSM to solve the inverse problem of a time-dependent heat source identification governed by a linear heat conduction equation. Kuo et al. [15] have provided a polynomial expansion method to solve the resultant three-point boundary value problems and obtained accurate heat source. Those papers were developed new techniques for the off-line identification of the unknown time-dependent heat source.

The remaining parts of this paper are arranged as follows. In Section 2, a novel view of the on-line identification problem of the unknown time-dependent heat source in a nonlinear heat conduction equation is presented as a set of nonlinear differential algebraic equations (DAEs) in the time domain. In Section 3 we give a new form of the nonlinear ordinary differential equations (ODEs), which allows us to derive an implicit $GL(n, \mathbb{R})$ Lie-group scheme in Section 4. Upon combining with the Newton iterative scheme we can derive a powerful numerical method, namely the Lie-group DAE (LGDAE) method, to solve the DAEs, which is convergent very fast. The numerical results obtained by applying the LGDAE to solve the real-time identification problem of time-dependent heat sources are presented in Section 5, where we test four linear heat conduction equations and three nonlinear heat conduction equations. We can observe that the present LGDAE can satisfy the above mentioned requirements. Finally, we draw some conclusions in Section 6.