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The Explicit Green's Approach with stability enhancement for solving the bioheat transfer equation

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ABSTRACT

The aim of this paper is to propose a strategy for performing a stability enhancement into the Explicit Green's Approach (ExGA) method applied to the bioheat transfer equation. The ExGA method is a time-stepping technique that uses numerical Green's functions in the time domain; these functions are here computed by the FEM. Basically, a new two nonequal time substeps procedure is proposed to compute Green's functions at the first time step. This is accomplished by adopting the standard explicit Euler scheme and an optimized procedure to yield the best stability constraint, allowing a reduction into the number of time steps without loss of accuracy. In addition, the concept of local numerical Green's functions is introduced and explored aiming at reducing the computational effort of nodal Green's functions calculation. Two examples are presented in order to show the potentialities of the proposed methodology, one to illustrate the accuracy and another applied to skin burn simulations.

1. Introduction

In recent years, there has been an increasing motivation in the development of mathematical models that describe heat transfer in living tissue with blood perfusion. Among many mathematical models of bioheat transfer [1,2], the Pennes' equation [3] is widely employed due to its simplicity and overall satisfactory representation of the physical phenomenon. Indeed, the Pennes' equation appears in a great number of bioheat applications such as hyperthermia, cryosurgery, hypothermia, thermography, skin burns, etc.

As the mathematical models and their applications in medical sciences are becoming more complex and multidisciplinary [1,2], numerical simulations are indispensible tools that partially replace laboratory testing, aiding in understanding the problem under consideration subjected to different inputs in an effective manner. In fact, over the years, the Pennes' bioheat transfer equation has been numerically solved by several numerical techniques such as the finite difference method [1,4–6], finite element method [1,7,8], boundary element method [9,10], meshless method [11,12] etc.

The objective of the present paper is to propose an improvement into the Explicit Green's Approach method [13–17] and apply it to solve the Pennes' equation. As an application of the Pennes' equation, the paper focuses on the numerical simulation of skin burns [1,6–8,22] that can be used to predict injury depths caused by different external heat supplies applied at the skin surface. The generality and success of the ExGA method relies heavily on numerical Green's functions rather than analytical ones employed in other formulations. It is well-known that Green's function methodologies are very powerful tools due to the ability to solve the problem under consideration subjected to different boundary conditions and heat source terms [18–24]. For instance, one can quote the work of Deng and Liu [19] where analytical Green's functions for the Pennes' equation that satisfy the homogeneous boundary conditions of the same problem were employed. However, these analytical Green's functions, even though very important to derive benchmark solutions, are not feasible in practice due to the difficulty of finding analytical expressions for Green's functions with arbitrary geometries and/or material properties. In this sense, the ExGA uses numerical Green's functions that also satisfy homogeneous boundary conditions, giving rise to a general time-integral expression that can easily handle any kind of geometry and medium. Furthermore, unlike time-domain BEM formulations in which analytical free-space Green's functions are generally employed [25,26], once Green's functions are computed in the ExGA method the solution is explicitly evaluated without the need of solving a system of equations.

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Nomenclature

c	specific heat (J kg^{-10} C^{-1})
$\mathbf{F}(t)$	external heat load vector
$\mathbf{G}(t)$	Green's matrix
$G(\mathbf{x}, \mathbf{y}, t-\tau)$	Green's function
$\mathbf{I}$	identity matrix
$\mathbf{K}$	conductivity matrix
$\mathbf{M}_L$	lumped capacity matrix
nq	number of equations
Q_m	metabolic heat of tissue (W m^{-3})
Q_r	spatial heating (W m^{-3})
t	time (s)
t_f	time of analysis
T_a	arterial temperature
$\mathbf{T}(t)$	time temperature vector

Greek symbols

Ω^h	discrete domain
$\Omega^h_{G_j}$	discrete local domain around $\mathbf{y}_j$
$\alpha_i \Delta t$	substeps
λ	eigenvalue
Δt	time step
ω_b	blood perfusion (s^{-1})
$\boldsymbol{\kappa}$	thermal conductivity tensor (W m^{-10} C^{-1})
Ψ	tissue damage
ρ	density (Kg m^{-3})
τ	time (or dummy) variable (s)

Superscript

k	time index

Subscript

b	blood

The main contributions of the present paper are the proposed use and optimization of a new nonequal two time substeps procedure to compute Green's functions at the first time step, a detailed discussion on the computation of local numerical Green's functions and a convergence analysis of the technique. The FEM in conjunction with the explicit Euler scheme are employed to compute Green's functions, represented by the so-called Green's matrix that stores numerical nodal values of the Green's functions. The nonequal time substeps values are calculated such that a maximum stability region for the time-integral expression of the ExGA method is achieved. Because of the discretization adopted, the numerical Green's function due to a point source possesses a compact support with values different from zero only in a small region around the source point and, as a consequence, its computation can be carried out locally in a straightforward manner.

The structure of the paper is organized as follows: in Section 2 the Pennes' bioheat transfer equation is briefly described and, in Section 3, a brief background is given on the theoretical foundations of the time-integral expression regarding the ExGA method discretized in a FEM sense. Section 4 presents the time substeps procedure to compute the Green's matrix by means of the explicit Euler scheme and a discussion about the convolution integral. Next, in Section 5, a detailed study of how to calculate the time substeps values by performing a stability and accuracy analysis is provided. In Section 6, two numerical examples are presented in order to assess the capabilities and potentialities of the improved ExGA method, including a convergence study and a simulation of skin burns caused by a heated plate. Finally, conclusions on the proposed methodology are drawn in Section 7.

2. Model equations

In this work, biological systems, more specifically skin tissue subjected to external factors leading to burns, are modeled by means of the Pennes bioheat transfer equation. Let the biological system (skin tissue) occupy an open and bounded domain $\Omega \subset \mathrm{R}^d$ where d is the order of space dimensions. The boundary of Ω is assumed to be sufficiently smooth and is denoted by $\Gamma = \partial\Omega$ with outward unit normal vector $\mathbf{n}$, $\bar{\Omega} = \Omega \cup \Gamma$ being the closure of Ω. In this way, the Pennes' bioheat transfer model can be formulated as: find the tissue temperature field $T : \bar{\Omega} \times [0, t_f] \to \mathbb{R}$ such that [1–3]:

$$\nabla \cdot (\boldsymbol{\kappa} \nabla T) + \omega_b \rho_b c_b (T_a - T) + Q_m + Q_r = \rho c \frac{\partial T}{\partial t} \quad \text{in } \Omega \times (0, t_f] \tag{1}$$

$$T = \bar{T} \quad \text{on} \quad \Gamma_D \times (0, t_f] \tag{2}$$

$$\boldsymbol{\kappa} \nabla T \cdot \mathbf{n} = \bar{q} \quad \text{on} \quad \Gamma_N \times (0, t_f] \tag{3}$$

$$\boldsymbol{\kappa} \nabla T \cdot \mathbf{n} = h(T_\infty - T) \quad \text{on} \quad \Gamma_R \times (0, t_f] \tag{4}$$

$$T = T_0 \quad \text{in} \quad \Omega \text{ at } \quad t = 0 \tag{5}$$

where the usual Dirichlet, Neumann and Robin (convective) type boundary conditions with their respective prescribed values and parameters are applied on Γ_D, Γ_N and Γ_R, respectively, such that $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ and $\Gamma_D \cap \Gamma_N = \Gamma_D \cap \Gamma_R = \Gamma_N \cap \Gamma_R = \varnothing$. In Eq. (1), T_a stands for the arterial temperature which is treated as a constant, $Q_m : \Omega \to \mathbb{R}$ and $Q_r : \Omega \times [0, t_f] \to \mathbb{R}$ denote the metabolic heat generation and the supplied heat source, respectively. The tissue properties are: $\boldsymbol{\kappa} : \bar{\Omega} \to \mathbb{R}^{d \times d}$ the thermal conductivity tensor, $\rho : \Omega \to \mathbb{R}$ the density and $c : \Omega \to \mathbb{R}$ the specific heat while the blood properties are: $\rho_b : \Omega \to \mathbb{R}$, $c_b : \Omega \to \mathbb{R}$ and $\omega_b : \Omega \to \mathbb{R}$ the blood perfusion. Finally, $T_0 : \Omega \to \mathbb{R}$ is the initial temperature field.

Due to the lack of a precise knowledge of the initial temperature in the whole biological system adopted in the model, in many numerical simulations the initial temperature $T_0(\mathbf{x})$ is set as the solution of a previous steady-state problem governed by the equation below

$$\nabla \cdot (\boldsymbol{\kappa} \nabla T_0) + \omega_b \rho_b c_b (T_a - T_0) + Q_m = 0 \quad \text{in} \quad \Omega \tag{6}$$

subjected to appropriate boundary conditions. For instance, in skin burns simulations, the initial temperature may be accomplished by considering that the skin surface is under a convective boundary condition, a situation normally encountered previous to the heating that causes burns.

3. Time stepping using the Green's matrix

The key feature of the ExGA method is the use of numerical Green's functions that satisfy homogeneous boundary conditions of the problem under consideration. Hence, according to