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A formalism for anisotropic heat transfer phenomena: Foundations and Green's functions

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ABSTRACT

A new complex-variable formalism for the analysis of three-dimensional (3D) steady-state heat transfer problems in homogeneous solids with general anisotropic behaviour is proposed in this paper. The derived method is based on the $\mathcal{R}$adon transform, which is used in order to reduce the dimension of the problem to a two-dimensional (2D) $\mathcal{R}$adon space where a solution can be easily handled via a complex-variable method. Subsequently, the 3D solution is obtained by applying the inverse $\mathcal{R}$adon transform. Despite that the main goal of this work is to develop and illustrate the general methodology, the proposed formalism is further applied to derive new Green's functions as application examples. Contributions include new forms for bimaterial and half-space Green's functions for line, point, vortex and dipole heat sources. In particular Green's functions due to heat vortex loop sources for infinite media, half-space and bimaterial systems are presented for *the first time*. The veracity and computability of the approach are demonstrated with some numerical examples.

1. Introduction

Many of the current applications in various branches of science and engineering use materials which exhibit anisotropic properties. Even for materials with overall isotropic response, in the current trend for the study of multiscale related problems, the incorporation of anisotropic features of their microconstituents is many times required. But anisotropy makes the differential operator of the governing equations more complicated than the well known isotropic case, mainly when cross-derivatives are involved [1]. This is particularly true for three-dimensional (3D) problems where the number of solutions is considerably minor than their corresponding two-dimensional (2D) case. This work is concerned with anisotropic or isotropic steady-state heat transfer problems, both 3D and 2D.

Most of the literature for anisotropic heat transfer phenomena is related to the coupled thermoelastic problem and restricted to 2D. Certainly the research devised for example by Clements [2], Wu [3], Sturla and Barber [4], Hwu [5], Yeh et al. [6] and Kattis et al. [7] deserve to be mentioned. These works may be considered as an extension of the Stroh formalism for elasticity to themoelasticity. Regarding the pure thermal problem, Chang et al. [8] present 2D and 3D closed-form fundamental solutions and; based on these solutions they solve the problem in a square, a circular disk, and an annular disk by using an integral-equation method. Employing Fourier transform Chang [9] derives steady and unsteady analytic Green's functions for a heat source in full-space (closed-form), half space (closed-form for 2D and integral-form for 3D), and infinite slab (series solution). Berger et al. also use Fourier transform to derive Green's functions due to a heat source for 2D bimaterial [10] and 2D and 3D functionally graded materials [11].

Complex-variable methods have also been used which dispense with performing inverse transforms. For instance Shen et al. [12] propose a series-form solution for the trimaterial system combining a 2D complex-variable method and analytic continuation. Such complex-variable methods are derived from thermoelastic studies (for instance the mentioned works [2–7]) and they are exclusively for 2D problems.

Alternative techniques use coordinate transformations which map the original problem with some particular material symmetries into an equivalent isotropic one. They have been widely used (see e.g. [13–17]), but they become quite cumbersome for

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Nomenclature

$(x_1, x_2, x_3), (s, p, x_3)$	Cartesian coordinate systems
$(\mathbf{n}, \mathbf{m}, \hat{\mathbf{e}})$	right-handed orthonormal triad
∂_i (i = 1,2,3)	differential operator $\frac{\partial}{\partial x_i}$
∂_s, ∂_p	differential operator $\frac{\partial}{\partial s}, \frac{\partial}{\partial p}$
$\check{f}, \mathcal{R}(f)$	$\mathcal{R}$adon transform of the function f
S_x	line
$\mathcal{L}(\partial_\mathbf{x})$	anisotropic differential operator
u	temperature
χ	potential function
k_{ij}	components of thermal diffusion tensor
q_i	components of heat flux
Q, R, T	real scalars
θ	angle
i	imaginary number
$A, B, C, D, \lambda, \kappa$	complex scalars
Λ_i, ξ_i, μ_i	components of complex vectors
$\breve{\boldsymbol{\omega}}$	vector
$\boldsymbol{N}$	matrix
ψ	holomorphic function
$\bar{z}$	conjugate of number z
$\overline{\psi}(z) = \overline{\psi(\bar{z})}$	conjugate function
$⨍$	Cauchy principal value
$\mathcal{B}(\cdot)$	$\mathcal{B}$ackprojection integral operator
$\mathcal{H}(f(p); p \to t)$	$\mathcal{H}$ilbert transform of the function f
$\tilde{f}(t) := -\frac{1}{4\pi}\mathcal{H}(\partial_p f(p); p \to t)$	
$\Re(\cdot)$	real part
$\Im(\cdot)$	imaginary part
$\delta(\cdot)$	Dirac delta function
δ_{ij}	delta de Kronecker
$\dot{f}$	first-order derivatives of f with respect to its argument
$\ddot{f}$	second-order derivatives of f with respect to its argument
$\mathbf{x}$	field point
$\mathbf{x}'$	source point
$\boldsymbol{\eta}, \boldsymbol{\zeta}$	spatial vectors
sgn(a)	1 if $a > 0$ and -1 if $a < 0$
UHP	upper half-plane
LHP	lower half-plane

multi-material problems [18], and the boundary conditions in problems involving several interfaces are not easy to handle.

As mentioned, for the 3D anisotropic case the works available in literature are limited. In addition to the already mentioned works by Chang et al. [8,9] and Berger et al. [11]; Mulholland and Gupta [19] investigated a procedure to solve arbitrary shape solids by using coordinate transformations to principal axes. Yen and Beck [20] use the Green's function to study the 3D problem in two-layered composite. The proposed solution results in double series in terms of eigenfunctions in two of the three directions. Recently, in the work by Marczak and Denda [21], a comprehensive review on full-space Green's functions (fundamental solutions) has been presented. In that work, 3D infinite Green's functions are derived by using the $\mathcal{R}$adon transform in order to reduce the problem to a one-dimension.

The domain-mapping techniques have been applied together numerical procedures like Boundary Element Methods (BEM) [22–25]; allowing to use schemes developed for isotropic materials. However simple and robust anisotropic Green's functions are welcome for the development of powerful boundary integral formulations and meshless methods [26–28].

Most of the above works study the temperature field when a heat (line or point) source is applied to the corresponding geometry domain. But another kind of sources can be possible. For potential problems, the concept of the field produced by a constant discontinuity of the potential field in a simple or double (dipole) layer appeared in literature several decades ago [29]. In the context of heat phenomena, the term *heat vortex* for a constant layer discontinuity in the temperature field has been introduced by Dundurs and Comninou [30]; and has been studied by Sturla and Barber [4] for anisotropic materials. Full-space, half-space and bimaterial Green's functions for heat lines and vortex could be obtained by analogy with the antiplane elastic problem [31] and corresponding dipole solutions could be derived by applying the limit procedure as described herein. However these Green's functions have not been found in our literature review. Furthermore, for solutions related to discontinuities in 3D there is a lack of studies considering different kind of sources, being this subject very important for the development of indirect BEM formulations.

The main goal of this work is to present a new complex-variable formalism for the analysis of general 3D anisotropic steady-state heat transfer problems based on a combination of $\mathcal{R}$adon transform and ideas from Stroh formalism. Since Green's functions play a fundamental role in finding partial differential equation solutions it is also the aim of this work to apply the derived methodology to obtain various 3D and 2D Green's functions. In order to illustrate the potential and validity of our approach Green's functions for some problems already analysed by other authors are first presented in a unified fashion. Subsequently the method is applied to derive new Green's functions not available into the scientific literature. In particular new 3D Green's functions for heat vortex loops in full-space, half-space and bimaterials.

The paper is outlined as follows. In Section 2, the detailed theory for 3D anisotropic heat transfer is derived. In this section, a 2D general solution is also introduced as a particularization of the well-known (vectorial-field) complex-variable Stroh formalism for anisotropic elasticity to the scalar-field elliptical problem that concerns us. The derived formalism is applied to obtain Green's functions for several kinds of singularities in Section 3. Validation and numerical results are presented in Section 4. The paper closes with a summary of the theory and concluding remarks in Section 5.

2. Theory of anisotropic heat transfer

2.1. Reduction of 3D Laplacian operator to a 2D $\mathcal{R}$adon space operator

Along the paper, it is assumed that Greek subindices range from 1 to 2 and Latin subindices from 1 to 3; moreover, repeated indices imply summation. The differential operator $\partial_i := \frac{\partial}{\partial x_i}$ is preferentially used; while the symbols ∂_p and ∂_s are reserved for $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial s}$, respectively. Consider a function $f(\mathbf{x})$ defined in a (x_i) $(i = 1, 2, 3)$ Cartesian coordinate system in 3D. For the fixed value x_3, the 2D $\mathcal{R}$adon transform in the plane with normal x_3 is defined by (see Appendix A, and references [32–34] for further details)

$$\check{f}(\mathbf{n}, p; x_3) := \int_{\mathbf{n}\cdot\mathbf{x}=p} f(\mathbf{x})\, \mathrm{d}S_x, \tag{1}$$

where $\mathbf{n} = (n_1, n_2, 0)^T$, with T denoting transpose, is a unit vector and S_x is a line perpendicular to $\mathbf{n}$ located at a (signed) distance p from the origin in the plane with normal x_3. Sometimes, the operator $\mathcal{R}(\cdot)$ defined by $\check{f} = \mathcal{R}(f)$ is also herein used to refer this transform.