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An analytical solution for the temperature field around a cylindrical surface subjected to a time dependent heat flux

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ABSTRACT

The analytical solution for the temperature field in an infinite solid medium which surrounds a cylindrical surface, determined by Carlslaw and Jaeger for the case of constant heat flux, is extended to the case of any time dependent heat flux. Then, with reference to a sinusoidally varying heat flux, the analytical solution is employed to determine benchmark results for the time evolution of the dimensionless temperature of the surface. These results are used to check the accuracy of the numerical solutions obtained by two different commercial codes: the finite-element code COMSOL Multiphysics and the finite-volume code FLUENT.

1. Introduction

An important transient heat conduction problem is the determination of the unsteady temperature field produced by a heated or cooled cylindrical surface surrounded by an infinite solid medium. An analytical solution of this problem, for the case of uniform and constant heat flux per unit area, has been presented by Carlslaw and Jaeger [1], and has been widely used for the design of Borehole Heat Exchanger (BHE) fields for ground-coupled heat pump systems. Indeed, the design method for BHE fields recommended by ASHRAE [2] and developed by Kavanaugh and Rafferty [3] is based on this solution. In fact, it sketches each BHE as a cylindrical heat source subjected to a uniform and constant heat flux per unit area, and considers the superposition of three heat pulses, each with a constant power, with different durations: 6 h, 1 month and 10 years. This method is still widely used, but recent studies showed that, if the seasonal heat loads are unbalanced and the effects of groundwater movement are negligible, large BHE fields can reach critical conditions after some decades [4,5], so that an analysis for a period of 10 years may be insufficient.

In the study of the long-term sustainability of BHE fields, time dependent heat loads must be considered. A time dependent heat load can be either obtained as a weighted superposition of constant heat loads displaced in time, or directly sketched by a suitable function of time. The second choice was performed, for instance, in Ref. [5], where the long-term performance of infinitely long BHE lines placed in a ground with negligible groundwater movement was studied numerically by means of the finite element software package COMSOL Multiphysics. Heat loads with period of 1 year were considered, varying with time with a sinusoidal law during each season, with different degrees of compensation of winter heating with summer cooling. The numerical code was validated in the special case of constant heat load, by comparison with the analytical solution of Carlslaw and Jaeger [1].

Indeed, an analytical solution for the temperature field around a cylindrical surface subjected to a uniform but time dependent heat flux and surrounded by an infinite solid medium would be useful, to validate the results obtained by means of numerical simulation codes, but is not directly available in the literature.

Two analytical solutions of the Fourier equation for the temperature field around a cylindrical surface placed in an infinite solid medium were presented, but both consider the case of a uniform and constant heat flux per unit area applied to the surface. The first is the well known solution by Carlslaw and Jaeger [1]; the second, reported in a book in German [6], is recalled in a more recent paper in English [7].

An analytical solution of the Cattaneo–Vernotte hyperbolic heat conduction equation in cylindrical geometry, with the boundary condition of a time variable heat flux at the inner surface, was determined by Barletta [8]. However, the author did not present or discuss the limit of his solution in the case of a vanishing relaxation time. Since the problem, especially in the hyperbolic heat conduction case, is very far from being elementary, the development of an independent solution of the Fourier equation, with the same geometry and boundary conditions, seems useful, to build a bridge between the simplest case, studied in Refs. [1,6],

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and the most complex case considered in Ref. [8], and to provide a cross validation of the available solutions.

In this paper, an analytical solution of the Fourier equation in an infinite solid medium bounded internally by a cylindrical surface subjected to any uniform and time-dependent heat flux, is presented. It is shown that the solution reduces to that determined by Carlslaw and Jaeger [1] in the special case of constant heat flux, and coincides with the limit for vanishing relaxation time of the solution determined by Barletta [8] for hyperbolic heat conduction. Then, for the special case of a sinusoidally varying heat flux, the analytical solution is employed to determine a benchmark table of values of the dimensionless temperature and to check the accuracy of numerical solutions obtained by two different commercial codes.

2. Analytical solution

Let us consider a homogeneous and infinite solid medium, bounded internally by the cylindrical surface $r = r_0$, i.e., which occupies the whole region of space $r_0 \leqslant r < +\infty$. Let us assume that the thermal conductivity k and the thermal diffusivity α of the medium are constants and no heat generation is present within the solid. At the initial instant of time, $\tau = 0$, the temperature field within the solid is uniform, with a value T_0, and stationary. For $\tau > 0$, a uniform and time-dependent heat flux per unit area $q(\tau) = q_0F(\tau)$ is applied to the solid, at the internal surface $r = r_0$, where $F(\tau)$ is a dimensionless function of time. Under these conditions, the temperature field in the solid is axisymmetric, and the differential equation for heat conduction in a cylindrical coordinate system can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}, \tag{1}$$

with initial and boundary conditions given by

$$T(r,0) = T_0, \tag{2}$$

$$-k\frac{\partial T}{\partial r}\bigg|_{r=r_0} = q_0F(\tau), \quad \tau > 0. \tag{3}$$

Let us introduce the dimensionless coordinates

$$\eta = \frac{r}{r_0}, \quad \xi = \frac{\alpha\tau}{r_0^2} \tag{4}$$

and the dimensionless temperature

$$\theta(\eta,\xi) = k\frac{T(r,\tau) - T_0}{q_0 r_0}. \tag{5}$$

Thus, Eqs. (1)–(3) can be rewritten as

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \theta}{\partial \eta} = \frac{\partial \theta}{\partial \xi}, \tag{6}$$

$$\theta(\eta,0) = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \eta}\bigg|_{\eta=1} = -\phi(\xi), \quad \xi > 0, \tag{8}$$

with

$$\phi(\xi) = F\left(\frac{\alpha\tau}{r_0^2}\right). \tag{9}$$

Eq. (6) can be solved by using the Laplace transform of $\theta(\eta,\xi)$ with respect to ξ,

$$\tilde{\theta}(\eta,s) = \int_0^{+\infty} e^{-s\xi}\theta(\eta,\xi)\mathrm{d}\xi. \tag{10}$$

In the Laplace transformed domain, Eq. (6) becomes

$$\frac{\partial^2 \tilde{\theta}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \tilde{\theta}}{\partial \eta} - s\,\tilde{\theta} = 0, \tag{11}$$

with the boundary condition

$$\frac{\partial \tilde{\theta}}{\partial \eta}\bigg|_{\eta=1} = -\tilde{\phi}(s). \tag{12}$$

The general solution of Eq. (11) is

$$\tilde{\theta}(\eta,s) = c_1(s)I_0(\eta\sqrt{s}) + c_2(s)K_0(\eta\sqrt{s}), \tag{13}$$

where $c_1(s)$ and $c_2(s)$ are arbitrary functions of s, while I_0 and K_0 are the modified Bessel functions of the first and second kind, with order 0. Since the dimensionless temperature must be vanishing for $\eta \to \infty$, the asymptotic properties of Bessel functions imply that $c_1(s) = 0$. By applying the boundary condition Eq. (12), and the relation $K'_0(z) = -K_1(z)$, we obtain the following solution of Eq. (11),

$$\tilde{\theta}(\eta,s) = \frac{\tilde{\phi}(s)K_0(\eta\sqrt{s})}{K_1(\sqrt{s})\sqrt{s}}, \tag{14}$$

where K_1 is the modified Bessel function of the second kind with order 1. On account of the convolution theorem, the inverse Laplace transform of Eq. (14) has the form

$$\theta(\eta,\xi) = \int_0^{\xi} \chi(\eta,u)\phi(\xi - u)\mathrm{d}u, \tag{15}$$

where

$$\chi(\eta,u) = L^{-1}\{A(\eta,s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{su}A(\eta,s)\mathrm{d}s, \tag{16}$$

$$A(\eta,s) = \frac{K_0(\eta\sqrt{s})}{K_1(\sqrt{s})\sqrt{s}}. \tag{17}$$

The integral in Eq. (16) can be evaluated by considering the closed path Γ shown in Fig. 1 as

$$\begin{aligned}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{su}A(\eta,s)\mathrm{d}s = \lim_{R\to+\infty}\lim_{\epsilon\to 0}\bigg\{ &\oint_{\Gamma} e^{su}A(\eta,s)\mathrm{d}s - \int_{\Gamma_R} e^{su}A(\eta,s)\mathrm{d}s \\ &- \int_{\Gamma_\epsilon} e^{su}A(\eta,s)\mathrm{d}s - \int_{AB} e^{su}A(\eta,s)\mathrm{d}s \\ &- \int_{CD} e^{su}A(\eta,s)\mathrm{d}s\bigg\}.\end{aligned} \tag{18}$$

Since there is no pole of $A(\eta,s)$ within the region bounded by Γ, the integral on Γ is zero, on account of Cauchy integral theorem. By considering the asymptotic expression of K_n, $K_n(s) \approx e^{-s}/\sqrt{2\pi s}$,

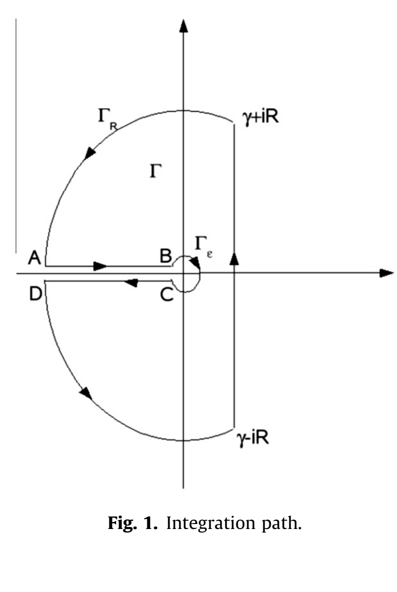

Fig. 1. Integration path.