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# Rare exact solution to a model of two-phase flow consisting of a nanofluid adjacent to a clear fluid

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ABSTRACT

We consider a theoretical model of a two-phase flow consisting of a nanofluid adjacent to a clear fluid. A method of obtaining the exact solution to the two-fluid vertical channel flow and convective heat transfer model is presented. The results constitute a rare case in which completely exact solutions are possible for a nonlinear flow problem involving nanofluids, as the governing equations are often highly nonlinear. We show rigorously that the nanofluid can modify the fluid velocity at the interface of the two fluids, and can be used to reduce shear at both the surface of the clear fluid and the interface of the two fluids. Upon exploring the existence of the exact solutions, we discover that in some situations there exist two mathematical solution branches, one of which is the physically relevant solution. We then discuss the behavior of the velocity and thermal profiles with the important parameters dictated by the nanoparticles.



## 1. Introduction

The effective thermal conductivity of nanofluids is expected to enhance the heat transfer properties of a fluid when compared with convectional heat transfer liquids, which suggests the possibility for the use of nanofluids in advanced nuclear systems Buongiorno and Hu [1]. Choi [2] was one of the first to use the term nanofluid to refer to a fluid with suspended nanoparticles. It has been shown [3,4] showed that the addition of a small amount of nanoparticles to convectional heat transfer liquids increases the thermal conductivity of the resulting fluid up to approximately two times. Buongiorno [5] has pointed out that nanoparticle absolute velocity can be viewed as the sum of the base fluid velocity and relative velocity (that he calls the slip velocity). Buongiorno considered in turn seven slip mechanisms: inertia, Brownian motion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and gravity settling. Numerous models and methods have been proposed to study the convective flows of nanofluids (see, for instance, [6–11]).

Recently Malashetty et al. [12,13] studied the free convective flow field and heat transfer of conducting two fluid flows in vertical and inclined channels. Van Gorder et al. [14] then extended this model in order to obtain a model which includes a nanofluid layer in such a two-fluid model. The framework was theoretical, as there are no known experimental two-layer fluid studies with nanoparticles. Hence, the results are mainly of theoretical interest. One possible application would be to flows in micro channels, where it would potentially be efficient to employ nanofluids. Furthermore, such results are valid for steady flows. Indeed, turbulent flows would induce mixing and other undesired behaviors. Hence, this framework would not be useful in describing such phenomenon as two-layer turbulent flows, or two-layer flows with mixing: a more complicated model would be required to take into account such behaviors.

In light of the numerical results of Van Gorder et al. [14], in the present paper we provide an exact solution to the two-fluid problem. Such a result not only verifies the numerical findings of Van Gorder et al. [14], but also is a rare instance of an exact solution to a nonlinear fluid flow problem. We start with a review of the physical problem in Section 2. In Section 3, we outline the mathematical problem which must be solved. In Section 4, we solve the nonlinear boundary value problem in general, obtaining an exact solution form which depends on coefficients which are in turn determined by model parameters. Using the obtained exact solutions, in Section 5 we discuss how to obtain physical parameters such as the Nusselt number and shear stress from the exact solutions, and we highlight certain results of interest. In Section 6, we offer conclusions which help to frame our results within the context of the physical problem. The results are discussed in the context of related studies [15–21].

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**Nomenclature**

| | |
|---|---|
| $C_p$ | specific heat at constant pressure |
| $C$ | nano particle volume fraction |
| $C_w$ | nano particle volume fraction at the left channel |
| $D_B$ | Brownian diffusion coefficient |
| $D_T$ | thermophoretic diffusion coefficient |
| $h$ | ratio of the width of the two regions |
| $h_i$ | width of the region I and II |
| $g$ | acceleration due to gravity |
| Gr | Grashof number |
| $K$ | ratio of thermal conductivities |
| $m$ | ratio of viscosities |
| $n$ | ratio of densities |
| $N_b$ | Brownian motion parameter |
| $N_t$ | thermophoresis parameter |
| Pr | Prandtl number |
| $p$ | pressure |
| $P$ | non-dimensional pressure gradient |
| $Q_i$ | heat generation/absorption coefficient of the region I and II |
| $Re$ | Reynolds number |
| $T$ | temperature |
| $T_{w_1}$ | temperature at the right wall |
| $T_{w_2}$ | temperature at the left wall |
| $u_i$ | velocities in the x-component of the regions I and II |
| $\tilde{u}_1$ | average velocity |
| $x$,$y$ | space coordinates |

*Greek letters*

| | |
|---|---|
| $\alpha_i$ | thermal diffusivity of the regions I and II |
| $\alpha$ | ratio of the thermal diffusivity |
| $\beta_i$ | coefficient of thermal expansion of the regions1 and 2 |
| $\beta$ | ratio of the coefficient of thermal expansion |
| $\mu_i$ | viscosities of the regions I and II |
| $\rho_i$ | densities of the regions I and II |
| $\theta_i$ | non-dimensional temperatures of the regions I and II |
| $\lambda$ | mixed convection parameter |
| $\phi$ | non-dimensional nanoparticle volume fraction |
| $\tau$ | heat capacity ratio |
| $v_i$ | kinematic viscosities of regions I and II |
| $\delta_i$ | non-dimensional internal heat generation/generation |
| $*$ | dimensionless quantity |

*Subscripts*

| | |
|---|---|
| 1 and 2 | refer to quantities for regions I and II, respectively. |

## 2. Physical formulation of the problem

For sake of completeness, we provide the problem formulation given in [14]. The physical configuration considered in this study is as shown in Fig. 1. The region $-h_2 \leqslant y \leqslant 0$ is occupied by a viscous incompressible nanofluid and the region $0 \leqslant y \leqslant h_1$ is occupied by a viscous, incompressible clear fluid (Newtonian). The two walls of the channel are held at different temperatures $T_{w_1}$ and $T_{w_2}$ with $T_{w_1} > T_{w_2}$. The flow in both the regions is assumed to be steady, laminar, fully developed, and the fluid thermo-physical properties are assumed to be constant except the density variation in the buoyancy term of the momentum equation (in both regions): This is called the Overbeck–Boussinesq approximation. Further, the fluids in both the regions are assumed to be driven by a common pressure gradient and the temperature difference at the walls. The basic two-fluid model of this situation (in the absence of nonofluids) was outlined in Malashetty et al. [12,13]. The model, as given in Van Gorder et al. [14] is

***Region I*** (Non-dimensionalized):

$$\frac{d^2u_1}{dy^2} - P + \lambda_1\theta_1 = 0, \tag{2.1}$$

$$\frac{1}{\Pr}\frac{d^2\theta_1}{dy^2} + \delta_1\theta_1 = 0, \tag{2.2}$$

***Region II*** (Non-dimensionalized):

$$\frac{d^2u_2}{dy^2} - h^2mP + h^2\beta mn\lambda_2\theta_2 = 0, \tag{2.3}$$

$$\frac{1}{\alpha}\frac{1}{\Pr}\frac{d^2\theta_2}{dy^2} + N_b\frac{d\theta_2}{dy}\frac{d\phi}{dy} + N_t\left(\frac{d\theta_2}{dy}\right)^2 + \delta_2\theta_2 = 0, \tag{2.4}$$

$$\frac{d^2\phi}{dy^2} + \frac{N_t}{N_b}\frac{d^2\theta_2}{dy^2} = 0. \tag{2.5}$$

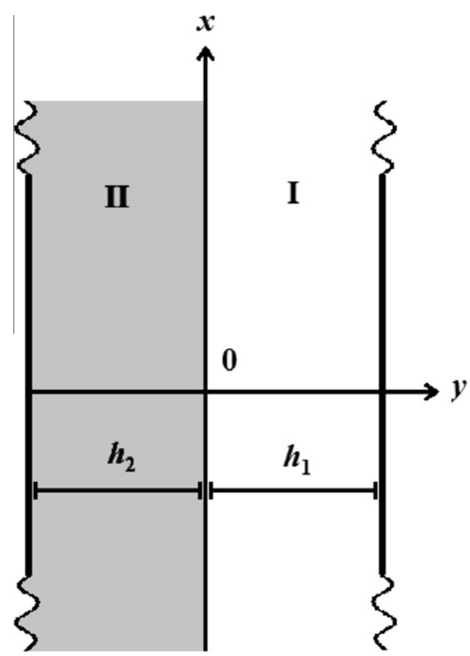


**Fig. 1.** Geometry of the problem. Region I contains the clear Newtonian fluid, while Region II contains the fluid with nanoparticles. The flow is vertical, directed along the $x$ axis (the direction, either toward positive or negative $x$, will depend on the fluid properties). The velocity and thermal properties vary in $y$, but are constant in $x$ (as the flow is assumed to be fully developed).

In order to account for the nanoparticle concentration, note that the form of the coupled Eqs. (2.4) and (2.5) accounting for the nanoparticle concentration are taken to follow from the theory outlined in Buongiorno [5]. In the above equations, the parameters $\lambda_1$, $\lambda_2$, Pr, $N_b$, $N_t$, $\delta_1$ and $\delta_2$ are the mixed convection parameters ($\lambda_1, \lambda_2$), Prandtl number, Brownian motion parameter, thermopherosis parameter, and heat source/sink parameter (in Regions I and II) respectively, and are defined by