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# A self-adaptive LGSM to recover initial condition or heat source of one-dimensional heat conduction equation by using only minimal boundary thermal data

Chein-Shan Liu *

*Department of Civil Engineering, National Taiwan University, Taipei, Taiwan*

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ABSTRACT

The present study is concerned with the recovery of an unknown initial condition for a one-dimensional heat conduction equation by using only the usual two boundary conditions of the direct problem for heat equation. The algorithm assumes a function for the unknown initial condition and derives an inverse problem for estimating a spatially-dependent heat source $F(x)$ in $T_t(x,t) = T_{xx}(x,t) + F(x)$. A self-adaptive Lie-group shooting method, namely a Lie-group adaptive method (LGAM), is developed to find $F(x)$, and then by integrations or by solving a linear system we can extract the information for unknown initial condition. The new method possesses twofold advantages in that no a priori information of unknown functions is required and no extra data are needed. The accuracy and efficiency of present method are confirmed by comparing the estimated results with some exact solutions.



## 1. Introduction

In this paper we consider an inverse problem of recovering an unknown initial condition for a one-dimensional heat conduction equation with only some minimal boundary conditions given:

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \ell, \ t > 0, \tag{1}$$

$$u(0,t) = u_0(t), \tag{2}$$

$$u(\ell,t) = u_\ell(t), \tag{3}$$

where the initial condition is absent as an unknown function of $x$, i.e.,

$$u(x,0) = f(x) = ? \tag{4}$$

For compatibility we require that $f(0) = u_0(0)$, and $f(\ell) = u_\ell(0)$. Besides the above two boundary conditions in Eqs. (2) and (3) for a typical direct problem of heat equation, the present method which will be developed in this paper does not need other data to recover $f(x)$.

In general, there are two classes of backward heat conduction problem (BHCP). For the first class it is usually by specifying a temperature distribution inside the domain of $0 \leqslant x \leqslant \ell$ at a certain instant of time $t = t_f$, and one wants to recover the initial temperature. Another class is that when there is no interior information about the temperature and only some boundary data are available one also attends to recover the initial temperature. However, in this case the existence of a solution for the inverse problem is not guaranteed [1].

In order to calculate the BHCP, there have appeared some progresses in this issue, including the boundary element method [2], the iterative boundary element method [3,4], the Tikhonov regularization technique [5,6], the operator-splitting method [7], the lattice-free high-order finite difference method [8], the contraction group technique [9], the fundamental solutions method [10,11], the third order mixed-derivative regularization technique [12], the Fourier regularization method [13], the three-spectral regularization methods [14], the regularization of Fredholm integral equation method [15], and the fictitious time integration method [16]. The method developed by Liu [9] was further developed by Xiong et al. [17], of which the stepsize used in the spatial finite difference is deemed as a regularization parameter. Chiwiacowsky and de Campos Velho [18] have given a review of the numerical solutions of the BHCP. Clark and Oppenheimer [19] and Ames et al. [20] have used a quasi-reversibility method to approximate the BHCP. The numerical implementation of the quasi-reversibility together with a time-direction Lie-group shooting method has been carried out by Chang et al. [21], which revealing a high performance for solving the BHCP.

Let

$$T(x,t) = u(x,t) - f(x) + 1, \tag{5}$$

* Tel.: +886 2 33664233; fax: +886 27396752.
*E-mail address:* liucs@ntu.edu.tw

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**Nomenclature**

| | |
|---|---|
| $\mathbf{A}$ | augmented matrix |
| $a$, $b$ | coefficients defined in Eqs. (29), (32), (38) and (50) |
| $C_j$ | a measure of convergence speed |
| $\mathbf{f}$ | $n$-dimensional vector field |
| $\hat{\mathbf{f}}$ | $:= \mathbf{f}(\hat{t}, \widehat{\mathbf{T}})$ |
| $\hat{f}_i$ | the $i$th component of $\hat{\mathbf{f}}$ |
| $f(x)$ | unknown initial temperature function |
| $F_0(t)$ | left-boundary function |
| $F_\ell(t)$ | right-boundary function |
| $\mathbf{F}$ | $:= \hat{\mathbf{f}}/\|\widehat{\mathbf{T}}\|$ |
| $F(x)$ | unknown heat source function |
| $F_i$ | $:=F(x_i)$ |
| $\mathbf{g}$ | $n+1$-dimensional Minkowski metric |
| $\mathbf{G}$ | an element of Lorentz group |
| $\mathbf{G}_i$, $i = 1,\ldots,K$ | elements of Lorentz group |
| $\mathbf{G}(r)$ | an element of Lorentz group |
| $\mathbf{G}(t_f)$ | an element of Lorentz group |
| $G_0^0$ | the 00th component of $\mathbf{G}$ |
| $\mathbf{I}_n$ | $n$-dimensional unit matrix |
| $\ell$ | length of rod |
| $\|\bullet\|$ | Euclidean norm |
| $\mathbb{M}^{n+1}$ | $n+1$-dimensional Minkowski space |
| $n$ | number of interior grid points |
| $q_0(t)$ | left boundary flux |
| $q_\ell(t)$ | right boundary flux |
| $r$ | weighting factor |
| $S$ | $:=t_f\|\mathbf{T}^f-\mathbf{T}^0\|$ |
| $SO_o(n,1)$ | $n+1$-dimensional Lorentz group |
| $so(n,1)$ | the Lie algebra of $SO_o(n,1)$ |
| $t$ | time |
| $t_f$ | final time |
| $\hat{t}$ | $:=(1-r)t_f$ |
| $\Delta t$ | time stepsize |
| $T$ | temperature |
| $\mathbf{T}$ | temperature vector of $T_i$ |
| $\mathbf{T}^0$ | initial temperature vector |
| $\mathbf{T}^f$ | temperature vector at final time $t_f$ |
| $\widehat{\mathbf{T}}$ | $:=r\mathbf{T}^0+(1-r)\mathbf{T}^f$ |
| $T_i(t)$ | $:=T(x_i,t)$ |
| $\widehat{T}_i$ | the $i$th component of $\widehat{\mathbf{T}}$ |
| $u_0(t)$ | left boundary temperature |
| $u_\ell(t)$ | right boundary temperature |
| $x$ | space variable |
| $x_i$ | discretized coordinate of $x$ |
| $\Delta x$ | mesh size of $x$ |
| $\mathbf{X}$ | $n+1$-dimensional augmented vector |
| $\mathbf{X}_k$ | numerical value of $\mathbf{X}$ at the $k$th time step |
| $\mathbf{X}^0$ | The value of $\mathbf{X}$ at initial time |
| $\mathbf{X}^f$ | The value of $\mathbf{X}$ at final time $t_f$ |
| $Z$ | $:=\exp(S/\eta)$ |

*Greek symbols*

| | |
|---|---|
| $\eta$ | coefficient defined in Eq. (35) |
| $\varepsilon$ | convergence criterion |
| $\theta$ | intersection angle of $\mathbf{T}^f-\mathbf{T}^0$ and $\mathbf{T}^0$ |

*Subscripts and superscripts*

| | |
|---|---|
| $i$ | index |
| $K$ | index |
| t | transpose |

where $f(x)$ is an unknown function of initial temperature to be determined. From Eqs. (1)–(4) we can derive

$$T_t(x,t) = T_{xx}(x,t) + F(x), \tag{6}$$
$$T(0,t) = F_0(t) = u_0(t) - u_0(0) + 1, \tag{7}$$
$$T(\ell,t) = F_\ell(t) = u_\ell(t) - u_\ell(0) + 1, \tag{8}$$
$$T(x,0) = 1, \tag{9}$$

where $F(x) = f''(x)$ is viewed as an unknown and spatially-dependent function of heat source.

In order to recover $f(x)$, Tadi [22] required overspecified boundary heat flux:

$$u_x(0,t) = q_0(t), \quad u_x(\ell,t) = q_\ell(t).$$

As compared with the above data required by Tadi [22], our requirement of measured data is minimal.

Mathematically speaking, Eqs. (6)–(9) form an underdetermined system, because both $F(x)$ and $T(x,t)$ are unknown functions. For the inverse problem of heat source identification there were many studies as can be seen from the papers by Cannon and Duchateau [23] for identifying $F(u)$, and Savateev [24] and Borukhov and Vabishchevich [25] for identifying $F(x,t)$ with additive or seperable space and time. Many researchers sought the heat source as a function of only space or time, for example, Farcas and Lesnic [26], Ling et al. [27], and Yan et al. [28].

Liu [29] and Yeih and Liu [30] have developed a two-stage Lie-group shooting method to estimate the time-dependent heat source. To the best knowledge of the author, in the open literature of inverse problems there has no researcher to discuss the possibility for estimating unknown heat source without needing of an extra measurement of data. We will develop a novel self-adaptive Lie-group shooting method, namely the Lie-group adaptive method (LGAM) for the inverse problem of *heat source identification* governed by Eqs. (6)–(9), as well as the *recovery of initial condition* in the BHCP governed by Eqs. (1)–(4).

Liu [31–33] has extended the group preserving scheme (GPS) developed previously by Liu [34] for ODEs to solve the boundary value problems (BVPs). In the construction of the Lie group method for the calculations of BVPs, Liu [31] has introduced the idea of one-step GPS by utilizing the closure property of the Lie group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM). After that, Liu [35] has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended the Lie-group method to estimate the thermophysical properties of heat conductivity and heat capacity [36–38]. The Lie-group method possesses a great advantage than other numerical methods due to its group structure, and it is a powerful technique to solve the inverse problems of parameters identification [39].

Liu and Atluri [40] have made a breakthrough for solving the Calderón's inverse problem by an effective combination of the Lie-Group Adaptive Method (LGAM) and the finite-strip technique. The LGAM views the Lie-group equation developed in the LGSM as a two-point Lie-group equation, describing a nonlinear relation between the state quantities defined at two different times or at two different positions of 1-D space. In this view we do not have a real target in the problem, and thus we can freely use the Lie-group equation as a supplemented equation, which is inherent in the ODEs, and thus we can solve many inverse problems by an iteration process. Very interesting, Liu [41] has applied the Lie-group adaptive method (LGAM) to identify the rigidity function of wave propagation problems without resorting on other data, besides those needed for the direct wave problem.