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Double-diffusive penetrative convection simulated via internal heating in an anisotropic porous layer with throughflow

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ABSTRACT

A model for double-diffusive convection in an anisotropic porous layer with a constant throughflow is explored, with penetrative convection being simulated via an internal heat source. Both linear instability and nonlinear stability analyses are performed to assess the suitability of linear theory to predict the destabilisation of the throughflow. It is shown that due to oscillatory instability modes, there are three distinct regions, where increasing the ascending throughflow from rest actually has a stabilising effect, before following the standard destabilisation and stabilisation pattern. This is a previously unobserved phenomenon. The agreement between the linear and nonlinear thresholds is substantial when a small descending throughflow is introduced, although this does deteriorate for ascending throughflow.

1. Introduction

The phenomenon of double-diffusive convection appears in numerous physical problems such as the spreading of pollutants, contaminant transport in saturated soil, underground disposal of nuclear wastes, and food processing, and has thus received extensive exploration in the literature, cf. [1]. The introduction of a throughflow leads to further important applications in hydrothermal porous systems [2], cloud physics [3,4], and in many industrial processes [1,5]. Recent research activity on throughflow has included [6–13], which have underlined the significant impact the presence of a throughflow has on the stability of a fluid/porous layer.

The motivation of this paper is to investigate both ascending, and descending, throughflow for double-diffusive convection in a fluid filled anisotropic porous layer with an internal heat source. Shivakumara and Khalili [10] studied this problem without the presence of an internal heat source, which we are introducing to allow penetrative convection to occur. Penetrative convection occurs when part of the layer has a tendency to be unstable which will then induce instability in the rest of the layer. Hill et al. [8] investigated a porous layer with throughflow, where the density is a nonlinear function of temperature to incorporate penetrative convection, and found that the stability behavior was significantly different for ascending and descending throughflow. This is in contrast to the case where no penetrative convection occurs [10].

Furthermore, the stability thresholds given in Hill et al. [8] are entirely monotonic, whereas in double-diffusive systems oscillatory behavior is usually present, which further motivates the exploration of this system. In fact, this paper shows that due to this behaviour there are three distinct regions, where increasing the ascending throughflow from rest has a stabilising effect, before following the standard destabilisation and stabilisation pattern (see e.g. [10]). This result has not been observed previously in the literature and is not present in the absence of an internal heat source [10]. In this paper we also consider the case where the porous medium is anisotropic, with constant anisotropic linearly layer dependent permeability [14].

When adopting a linear analysis approach, the perturbation to the steady state is assumed to be small, and so nonlinear terms in the governing set of partial differential equations are discarded. It has been proved that linear analysis often provides little information on the behavior of the nonlinear system [15], so in such cases only instability can be deduced from the linear thresholds, as any potential growth in the nonlinear terms is not considered.

In order to establish stability results we turn our attention to the highly adaptable energy method [15]. Nonlinear energy methods are particular useful as they delimit the parameter region of possible subcritical instability (the region between the linear instability and nonlinear stability thresholds) [16]. Hence, quantifying the discrepancy between these two thresholds makes it possible to provide an assessment of the suitability of linear theory to predict the de-stabilisation of the throughflow.

The solutions of these two theories reduce to generalised eigenvalue problems which have been derived numerically using the Chebyshev-tau technique [17,18]. Standard indicial notation is employed, and fixed boundary conditions are taken throughout.

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2. Formation of the problem

Let us consider a water saturated porous layer Ω_p bounded by two horizontal parallel planes. Let $d > 0$, $\Omega_p = \mathbb{R}^2 \times (0, d)$ and $Oxyz$ be a cartesian frame of reference, with standard indicial notation employed throughout the paper. The Darcy equation, for variable permeability $k(z) = k_0 s(z)$, is assumed to govern the fluid motion in the layer, such that

$$0 = -p_{,i} - \frac{\mu}{k(z)} v_i - b_i g \rho, \tag{1}$$

where v_i and p are velocity, and pressure, $\mathbf{b} = (0, 0, 1)$, g is acceleration due to gravity, μ is the dynamic viscosity of the fluid and k_0 is the reference permeability. Denoting T to be the temperature, and C to be the concentration of the dissolved species, the density $\rho(T, C)$ is given by

$$\rho(T, C) = \rho_0(1 - \alpha_t(T - T_0) + \alpha_c(C - C_0)),$$

where ρ_0, T_0, and C_0 are reference values of density, temperature and concentration respectively, and α_t and α_c are the coefficients for thermal and solutal expansion.

Eq. (1) together with the incompressibility condition, and the equations of energy and solute balance, yield the following system of governing equations

$$\begin{aligned} &0 = -p_{,i} - \frac{\mu}{k_0 s(z)} v_i - b_i g \rho_0(1 - \alpha_t(T - T_0) + \alpha_c(C - C_0)),\\ &v_{i,i} = 0,\\ &\frac{1}{M} T_{,t} + v_i T_{,i} = \kappa_t \nabla^2 T + Q,\\ &\phi C_{,t} + v_i C_{,i} = \kappa_c \nabla^2 C. \end{aligned} \tag{2}$$

In these equations ϕ is the porosity and κ_c is the solute diffusivity, with

$$(\rho_0 h)_m = (1 - \phi)(\rho_0 h)_s + \phi(\rho_0 h_p)_f,$$

where h_p is the specific heat of the fluid, and h is the specific heat of the solid, with the subscripts f, s, and m referring to the fluid, solid and porous components of the medium, respectively. Given that κ is the thermal diffusivity, the remaining terms of system (2) are defined as $\kappa_t = \kappa/(\rho_0 h_p)_f$ and $M = (\rho_0 h_p)_f/(\rho_0 h)_m$.

The $Q > 0$ term in (2) is some (constant) internal heat source and its inclusion allows the model to describe penetrative convection in the porous layer. The boundary conditions for the problem are $v_i = 0$, $T = T_U$ and $C = C_U$ at $z = d$ and $v_i = 0, T = T_L$ and $C = C_L$ at $z = 0$, where $C_L > C_U$, so that the system is being salted from below. We allow for the two cases of heating from below $T_L > T_U$ and from above $T_L < T_U$.

Let us now consider the basic steady state solution $(\bar{v}_i, \bar{p}, \overline{T}, \overline{C})$ of (2), with a throughflow in the z direction of the form

$$\bar{\mathbf{v}} = (0, 0, V),$$

where V is constant. Utilising the boundary conditions, Eqs. $(2)_3$ and $(2)_4$ show that

$$\overline{T}(z) = \frac{Qz}{V} + T_L + \frac{V(T_L - T_U) + Qd}{V\left(e^{\frac{Vd}{\kappa_t}} - 1\right)}\left(1 - e^{\frac{Vz}{\kappa_t}}\right),$$

$$\overline{C}(z) = C_L + \frac{C_L - C_U}{1 - e^{\frac{Vd}{\kappa_c}}}\left(e^{\frac{Vz}{\kappa_c}} - 1\right).$$

In contrast to the classical Bénard problem, the steady temperature field for this problem is clearly not linear in z. A derivation of the hydrostatic pressure $\bar{p}$ may be found from $(2)_1$, but is not included as it is eliminated in subsequent analyses.

To assess the stability of the steady solution we introduce a perturbation (u_i, θ, c, π) to the steady state solution, such that $v_i = \bar{v}_i + u_i$, $T = \overline{T} + \theta$, $C = \overline{C} + c$, $p = \bar{p} + \pi$, and non-dimensionalise with scalings of

$$u_i = \frac{\kappa_t}{d} u_i^*, \quad \pi_i = \frac{\mu\kappa_t}{k_0}\pi_i^*, \quad \theta = \theta^* \sqrt{\frac{dQ\mu}{g\rho_0\alpha_t k_0}}, \quad x_i = d x_i^*,$$

$$c = c^* \sqrt{\frac{\mu\kappa_t(C_L - C_U)}{g\rho_0\alpha_c k_0 d}}, \quad t = \frac{d^2}{\kappa_t M} t^*.$$

Substituting the perturbations and non-dimensionalised variables into system (2), and dropping the stars we derive

$$\begin{aligned} &0 = -\pi_{,i} - \frac{1}{f(z)} u_i + b_i R\theta - b_i R_s c,\\ &u_{i,i} = 0,\\ &\theta_{,t} + u_i\theta_{,i} + T_f \theta_{,z} = RG(z)w + \nabla^2\theta,\\ &\hat{\phi} c_{,t} + u_i c_{,i} + T_f c_{,z} = R_s M(z) w + \frac{1}{Le}\nabla^2 c, \end{aligned} \tag{3}$$

where $w = u_3$, $f(z) = s(z/d) = 1 + \lambda z$ with $\lambda > -1$ to ensure $f(z) > 0$,

$$G(z) = \frac{T_f}{e^{T_f} - 1}\left(\varepsilon + \frac{1}{T_f}\right)e^{T_f z} - \frac{1}{T_f},$$

$$M(z) = \frac{Le T_f e^{Le T_f z}}{e^{Le T_f} - 1},$$

and

$$\hat{\phi} = M\phi, \quad T_f = \frac{Vd}{\kappa_t}, \quad \varepsilon = \frac{(T_L - T_U)\kappa_t}{Qd^2}, \quad Le = \frac{\kappa_t}{\kappa_c},$$

$$R^2 = \frac{g\rho_0\alpha_t k_0 d^3 Q}{\mu\kappa_t^2}, \quad R_s^2 = \frac{g\rho_0\alpha_c k_0 d(C_L - C_U)}{\mu\kappa_t},$$

with R^2 and R_s^2 being the thermal and solute Rayleigh numbers, respectively and T_f being the non-dimensional form of the throughflow. It is important to note that $\varepsilon > 0$ and $\varepsilon < 0$ correspond to heating from below and above, respectively.

The perturbed boundary conditions are now $u_i = \theta = c = 0$ at $z = 0, 1$. We assume that the perturbation fields (u_i, θ, c, π), defined on $\mathbb{R}^2 \times [0, 1]$, are periodic in the x and y direction and we shall denote by $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ to be the periodicity cell.

Due to the specific non-dimensionalisation used, as the throughflow V tends to 0 (which is equivalent to $T_f \to 0$), all singularities are removable, allowing system (3) to revert to the standard classical system where no throughflow is present. This follows as $M(z) \to 1$ and $G(z) \to \varepsilon + z - 1/2$ as $T_f \to 0$.

3. Linear instability analysis

To proceed with the linear analysis the nonlinear terms from (3) are discarded. Since the resulting system is linear and autonomous we may seek solutions of the form $u_i = e^{\sigma t}u_i(\mathbf{x})$, $\theta = e^{\sigma t}\theta(\mathbf{x})$, $c = e^{\sigma t}c(\mathbf{x})$, and $\pi = e^{\sigma t}\pi(\mathbf{x})$, where σ is the growth rate and a complex constant. Taking the double *curl* of the linearised version of $(3)_1$, using the third component, (and the fact that $\mathbf{u}$ is solenoidal) we have the linearised system

$$\begin{aligned} &\nabla^2 w - \frac{f_{,3}}{f} w_{,3} - Rf\nabla_1^2\theta + R_s f \nabla_1^2 c = 0,\\ &\nabla^2\theta - T_f\theta_{,3} + RGw = \sigma\theta,\\ &\frac{1}{Le}\nabla^2 c - T_f c_{,3} + R_s M w = \hat{\phi}\sigma c, \end{aligned} \tag{4}$$

where $\nabla_1^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. We now introduce normal modes of the form $w = w(z)p_f(x, y)$, $\theta = \theta(z)p_f(x,y)$ and $c = c(z)p_f(x,y)$, where $p_f(x,y)$ is a plan-form which tiles the plane (x, y) with $\nabla_1^2 p_f = -a^2 p_f$, such that $a^2 = a_x^2 + a_y^2$. The plan-forms represent