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Time evolution of the heat diffusion phenomenon from the point source near the interface

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A B S T R A C T

The characteristic of the mathematical solution describing linear heat diffusion from an instantaneous point source near the interface of two semi-infinite domains in perfect thermal contact is discussed. Because spherical symmetry is lacking and the solution cannot be scale-transformed into a time-independent shape, the only way to grasp the geometrical information of the solution is by ad-hoc computation. The solution which was first derived more than 65 years ago is re-arranged to permit physical interpretation. Through the term-by-term analysis, the time evolution of the solution is divided into three stages and this enables systematic understanding of its nature.

1. Introduction

Diffusion is a universal phenomenon widely observed for particles, heat, momentum, etc. It is the macroscopic spontaneous transfer induced by microscopic random motion. When the microscopically observed visual field is close to μm or less, diffusional transfer could be dominant. As one example, we offer the living cell which is considered to be the fundamental life unit. Inside a 10 μm diameter space, coexisting with abiotic diffusion, macroscopic vital functions emerge [1]. Innovations in molecular biology and various measurement technologies have enabled the direct observations of intracellular transfer and this now requires deeper knowledge of diffusional phenomena at the cell interface.

Not only the mathematical context of the initial-boundary value problem, but also its wide application area, has attracted the interest of researchers for many years to the diffusion problem at the interface. Some exact analytical solutions have been derived for different interface models; but all have met an obstacle because of their mathematical complexity. Recent growth of the calculation capability of computers helps to overcome these difficulties, but we still need analytical solutions to understand the physics behind the phenomena.

One of the approaches to derive the analytical solution for the problem which is difficult to solve exactly, is to develop a suitable approximation method. The heat-balance integral method (HBIM) can systematically derive an approximate analytical solution, including nonlinear problems like change of phase [2]. Two other mathematical techniques include the double-integration method (DIM) [3] and the refined integral method [4]. Recently, studies have been made on the integral-balance approaches [5-7] and the multiple integral-balance method (MIM) [8] was proposed which generalized the HBIM and DIM .

If we limit the problem to a linear diffusion equation, the solution against the impulse input (the fundamental solution) can be used to construct a wide variety of initial boundary conditions by the convolution integral [9]. Therefore, our primary task is to comprehend the characteristics of the fundamental solution.

For a linear diffusion problem for two neighboring semi-infinite media, Sommerfeld [10] derived a one-dimensional exact solution. In 1949 Bellman, Marshak, and Wing (B.M.W.) [11] derived a three-dimensional exact solution for neutron aging. As described in the next section, the latter solution which is referred to as the B.M.W. solution, has a complicated form. Although some adjustments and contractions were tried with the intention of treating heat conduction problems [12], it was still difficult to get a physical interpretation directly from the B.M.W. solution.

Shendeleva [13-15] has been working on a derivation of the solution for two and three-dimensional problems by two different approaches for heat diffusion and light propagation problems. She proposed a novel wave diagram method in order to get a visual image of the solution. As analogies with light ray, wave front, and optical distance in the field of optics, she introduced the concept of the ray of a temperature field, thermal wave front, and thermal distance. This approach provides an intuitive way to get a wavelike image of heat diffusion near the interface.

Nonetheless, the ray of a temperature field is not equal to heat flux and a thermal wave front is not equal to an isothermal surface. We still need

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many computational trials to get a temperature distribution, thermal flow and the time evolution of the distribution and flow. We need to elicit the interpretation which can fill the gap between the wave diagram method and an ad-hoc numerical computation from the B.M.W solution.

We start from a re-arrangement of the B.M.W. solution in order to clarify physical role, term-by-term. As a result, the existence of the time region in which the nature of mutual similarity vanishes will be indicated. Before and after this time region, the mutual similarity is maintained and it is easy to understand the geometric features of the solution systematically.

2. Model and solution

Our problem and variables are shown in Fig. 1. Two semi-infinite domains are bordering on $z = 0$ in perfect thermal contact. An instantaneous point heat source with q [J] energy is set at $(x, y, z) = (0,0, z')$, $(z' > 0)$. Both domains have different thermal properties, but are assumed to be isotropic, uniform and constant, respectively. We ignore convection and radiation.

This problem can be expressed as an initial boundary value problem of the heat conduction equation system given below.

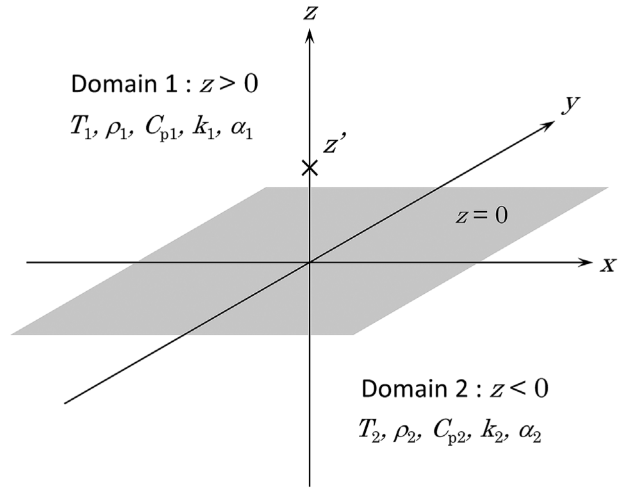

Fig 1. Model for our heat conduction problem. T[K]: Temperature, ρ[kg/m^3]: Density, C_p[J/(kg × K)]: Specific heat capacity, k[W/(m × K)]: Thermal conductivity, α[m^2/s]: Thermal diffusivity, $z' > 0$: Position of the point heat source. The subscript designates its domain.

$$
\begin{aligned}
&\rho_1 C_{p1}\frac{\partial T_1}{\partial t} = k_1\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} + \frac{\partial^2 T_1}{\partial z^2}\right), \quad z > 0\\
&\rho_2 C_{p2}\frac{\partial T_2}{\partial t} = k_2\left(\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} + \frac{\partial^2 T_2}{\partial z^2}\right), \quad z < 0\\
&\lim_{t\searrow 0} T_1 = \frac{q}{\rho_1 C_{p1}}\delta(x)\delta(y)\delta(z - z'), \quad \lim_{t\searrow 0} T_2 = 0\\
&\lim_{z\searrow 0} T_1 = \lim_{z\nearrow 0} T_2, \quad \lim_{z\searrow 0} k_1\frac{\partial T_1}{\partial z} = \lim_{z\nearrow 0} k_2\frac{\partial T_2}{\partial z}
\end{aligned}
\tag{1}
$$

The solution for each domain can be described as below [12].

$$
T_1(x, y, z, z', t) = \frac{q}{\rho_1 C_{p1}} \times \frac{1}{(4\pi\alpha_1 t)^{\frac{3}{2}}}\left\{\exp\left(-\frac{r_-^2}{4\alpha_1 t}\right) - \exp\left(-\frac{r_+^2}{4\alpha_1 t}\right)\right\} + \frac{q}{\rho_1 C_{p1}} \times \frac{\beta^2}{8\pi^2(\alpha_1 t)^{\frac{3}{2}}}\int_0^1 \frac{\exp\left\{-\frac{\beta^2 R^2}{4(\beta^2 u + 1 - u)}\right\}}{(\beta^2 u + 1 - u)\sqrt{u(1-u)}} f_1(Z, Z', \sigma, u)du \tag{2}
$$

$$
f_1(Z, Z', \sigma, u) = \frac{\sigma^3 u(Z' - Z)}{(\sigma^2 u + 1 - u)^2}\exp\left\{-\frac{(Z' - Z)^2}{4u}\right\} + \frac{\sigma\sqrt{\pi u(1-u)}}{(\sigma^2 u + 1 - u)^{\frac{3}{2}}}\left\{1 - \frac{(Z' - Z)^2\sigma^2}{2(\sigma^2 u + 1 - u)}\right\}\exp\left\{-\frac{(Z' - Z)^2\sigma^2}{4(\sigma^2 u + 1 - u)}\right\}\mathrm{erfc}\left\{\frac{(Z' - Z)(1-u)}{2\sqrt{u(1-u)(\sigma^2 u + 1 - u)}}\right\} \tag{3}
$$

$$
T_2(x, y, z, z', t) = \frac{q}{\rho_1 C_{p1}} \times \frac{\beta^2}{8\pi^2(\alpha_1 t)^{\frac{3}{2}}}\int_0^1 \frac{\exp\left\{-\frac{\beta^2 R^2}{4(\beta^2 u + 1 - u)}\right\}}{(\beta^2 u + 1 - u)\sqrt{u(1-u)}} f_2(Z, Z', \sigma, u)du \tag{4}
$$

$$
f_2(Z, Z', \sigma, u) = \frac{\beta Z(1-u) + \sigma^3 u Z'}{(\sigma^2 u + 1 - u)^2}\exp\left\{-\frac{Z'^2(1-u) + \beta^2 Z^2 u}{4u(1-u)}\right\} + \frac{\sigma\sqrt{\pi u(1-u)}}{(\sigma^2 u + 1 - u)^{\frac{3}{2}}}\left\{1 - \frac{(\beta Z - \sigma Z')^2}{2(\sigma^2 u + 1 - u)}\right\} exp\left\{-\frac{(\beta Z - \sigma Z')^2}{4(\sigma^2 u + 1 - u)}\right\} erfc\left\{\frac{(1-u)Z' + \sigma\beta Z u}{2\sqrt{u(1-u)(\sigma^2 u + 1 - u)}}\right\}\# \tag{5}
$$

The abbreviated variables are as listed below.

$$
\begin{aligned}
&r^2 = x^2 + y^2, \quad r_-^2 = r^2 + (z - z')^2, \quad r_+^2 = r^2 + (z + z')^2,\\
&R = \frac{r}{\sqrt{\alpha_1 t}}, \quad Z = -\frac{z}{\sqrt{\alpha_1 t}}, \quad Z' = \frac{z'}{\sqrt{\alpha_1 t}}, \quad \beta = \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}, \quad \sigma = \frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}
\end{aligned}
\tag{6}
$$

Expressions (2) to (6) refer to the Carslaw and Jaeger expression [5] which was arranged from the original B.M.W. expression. The integral which was introduced during the inverse Laplace transformation for the products of two functions [11] cannot be symbolically solved. If we try to apply the B.M.W. solution, numerical computation cannot be avoided, but the physical interpretation may enable our much better understanding about diffusion near an interface.

3. Re-arrangement and term-by-term analysis of the B.M.W. solution

Our algebraic operations are elementary, but rational from the physical viewpoint. The basic objective is to clarify the origin of the terms outside the exponential and complementary error functions. The drawback of the re-arrangement is the eventual lengthy equation. In order to compensate for the loss of algebraic compactness, the solution is divided into some blocks and labelled exteriorizing their features. The naming convention for the functions defined here is explained briefly in Fig. 2.