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A modified fractional order generalized bio-thermoelastic theory with temperature-dependent thermal material properties

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A B S T R A C T

In present work, the theory of a modified fractional order generalized bio-thermoelasticity with variable thermal material properties is developed. To keep the dimensions of fractional order thermoelastic theory consistent, a new parameter is first introduced into the generalized fractional order heat conduction equation. One-layered skin tissue with variable thermal properties is used for the numerical evaluation after the accuracy of the modified fractional order bio-thermoelastic model is verified. The effects of fractional order parameter, the added parameter and temperature-dependent thermal properties on the responses of skin tissue are discussed and illustrated graphically.

1. Introduction

With the rapid development of laser, microwave, radio-frequency and focused ultrasound, a number of modern thermo-therapeutics have been widely used in clinical treatment. One of the biggest challenges in thermal therapy is delivering the appropriate heat energy to the diseased tissue without affecting the healthy tissue. Thus it is important to understand how the temperature/stress fields affect the kinetics in the thermal treatment.

It is noted that even a small change of heat-induced stress can suppress immune response, alter production of hormones and protein denaturation [1]. However, most studies mainly focus on the heat conduction [2–11], the heat induced deformation is not considered. Based on the Pennes' bioheat transfer equation [2], Shen et al. [12] studied the static thermo-mechanical responses of skin tissue at high temperature. Xu et al. [13,14] investigated the heat transfer, thermal damage and heat-induced stress of human skin. Kim et al. [15] analyzed the transient thermal-mechanical responses of innocuous tactile stimulation induced by laser. Nevertheless, it can be found that the mechanical behavior has no effect on the distribution of temperature in these studies.

It is well known that the classical uncoupled and coupled theories of thermoelasticity predict [16] an infinitely fast propagation of thermal signal, which contradict physical facts [17]. To eliminate such paradox, a number of generalized thermoelastic theories involving a finite speed of heat conduction have been proposed. Lord and Shulman [18] developed the generalized thermoelastic theory with one relaxation time by using a wave-type heat conduction law to replace Fourier's heat conduction law. Green and Lindsay [19] introduced the temperature rate into the constitutive equations and developed a thermoelastic theory with two relaxation times. Green and Naghdi [20–22] proposed a theory based on three types constitutive equations, which labeled as G-N I, II, III. When the theory is linearized, G-N I is equivalent to Fourier's heat conduction law; G-N II predicts that heat propagates at a finite speed and involves no energy dissipation; G-N III includes a thermal damping term and thermal wave tends to diffusive with the increasing of damping coefficient.

However, the classical and generalized thermoelastic theories fail to accurately predict the temperature and stress of materials, such as amorphous media, glassy, porous material, man-made and biological materials/polymers when the media subject to cryogenic temperature or transient thermal loading. Recently, fractional calculus has been applied in diverse fields, including physics, chemistry, biology, hydrology and mathematical finance. One of the main reasons for its popularity is that it has memory characteristic, which provides a natural setting for describing various transport processes in the complex anisotropic and non-homogeneous media. Abel [23] first applied fractional calculus to solve the tautochrone problem. The good agreement with experimental results can be obtained when using fractional derivatives to describe the viscoelastic materials [24–26]. Then lots of physical models are developed in the context of fractional calculus, such as heat conduction, diffusion, viscoelasticity and electricity [27–31]. The existence and uniqueness of the solutions of fractional differential equations have been verified in many theoretical studies

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Nomenclature

S	Entropy density of skin tissue
q_i	Components of heat flux vector (W/m^2)
c_b	Specific heat of blood (J/kg K)
k_{ij}^*	Material parameter of G-N theory
c	Specific heat of skin (J/kg K)
c_0	Constant specific heat at reference temperature (J/kg K)
k_0^*	Constant material parameter at reference temperature
c_{ijkl}	The elastic stiffness
u_i	Components of displacement vector (m)
χ	Small quantity shows the influence of temperature (K^{-1})
e_{ij}	Components of strain tensor
e	The cubical dilatation
t	Time (s)
f_i	Body force

Greek symbols

T	Absolute temperature (K)
θ	Temperature increment (K)
w_b	Blood perfusion rate (s^{-1})
σ_{ij}	Components of stress tensor (pa)
T_b	Blood temperature (K)
δ_{ij}	Kronecker delta function
Q_{met}	Metabolic heat generation (W/m^3)
ρ_b	Blood mass density (kg/m^3)
τ	The new added parameter (s)
ρ	Skin tissue mass density (kg/m^3)
α	Fractional order parameter
ϑ	Kirchhoff transformation of θ
γ_{ij}	The thermal constants
λ_{xx} and μ_{xx}	Lame's constants (kg/ms^2)
T_0	Reference temperature, $T_0 = 310$K
α_t	Thermal expansion coefficient (K^{-1})
$\Gamma(\alpha)$	The gamma function

Subscript

x, y, z	Space coordinate
$i,\ j,\ k$	Number of space domain

[32–34]. Diethelm [35] proposed the numerical algorithms to deal with engineering problems in the context of fractional calculus. Fractional heat conduction in a composite metal medium was investigated by Povstenko [36]. Povstenko [37,38] also proposed the quasi-static uncoupled fractional order classical thermoelasticity and the thermoelasticity without energy dissipation by replacing the first-order time derivative with a derivative of arbitrary positive real order α. The Cattaneo-type space-time fractional heat conduction and diffusion equations were built by Qi and Jiang [39], they obtained the analytical solutions of the Cauchy problem. Ezzat et al. [40] obtained the thermal behavior in biological tissue with the fractional form of Pennes' bioheat transfer equation. Jiang and Qi [41] derived a thermal wave model of bioheat transfer with Riemann-Liouville fractional derivative induced by spatial heating. The time-fractional dual phase lag bioheat transfer equation was built by Xu and Jiang [42].

In the above work, the material properties of biological tissue are taken to be constant. However, at high temperature, thermal properties are no longer constant and change with temperature [43–45]. The temperature-dependent thermal properties in materials have become very important in engineering, such as drying of porous solids, nuclear fusion process, etc. Godfrey [46] has reported that the thermal conductivity of ceramic would decrease 45% when the temperature increased from 1 to 400°C. Lakhssassi et al. [47,48] and Tunc et al. [49] studied heat conduction problems with variable thermal conductivity in the context of Pennes' bioheat transfer equation.

So far, few attempts are made to solve the fractional order bio-thermoelastic coupling problem with temperature-dependent thermal properties, even if there are studies of variable thermal properties only limited to heat conduction [47–49]. It should be noted that most existing studies about using the fractional derivative to replace the integer order derivative directly but do not consider the fact that this substitution will result in dimensionally inconsistent [36–38,50–55]. In present work, we aim to solve this problem of dimensionally inconsistent in fractional heat conduction theory [36–38] and develop a new fractional order generalized bio-thermoelastic theory. The effects of temperature-dependent thermal properties and fractional order parameter on the distributions of temperature, displacement and stress are discussed and represented graphically.

2. Governing equations

It is known that the skin tissue is a highly complex and anisotropic structure. In this section, we aim to modify the generalized fractional heat conduction law of G-N II model [36–38] and formulate a new fractional order generalized bio-thermoelastic model for the anisotropic skin tissue.

Analogy to Povstenko [36], the fractional order generalized heat conduction law of G-N II model takes the following form:

$$\frac{\partial q_i}{\partial t} = -k_{ij}^* I^{\alpha-1}\theta_{,j} \quad (0 < \alpha \le 2) \tag{1}$$

where $\theta = T - T_0$ is temperature increment and $|\theta/T_0| \ll 1$; $I^{\alpha-1}$ is the Rieman-Liouville fractional integral operator, and

$$I^{\alpha-1}f(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}f(\tau)d\tau \quad (0 < \alpha \le 2)$$
$$I^0 f(t) = f(t), \quad I^{-\alpha}f(t) = \frac{\partial^\alpha}{\partial t^\alpha}f(t) \tag{2}$$

In the present work, comma followed by sub-index denotes the corresponding partial differentiation. Youssef [51] proposed the physical meaning of the fractional order parameter: $0 < \alpha < 1$ indicated weak conductivity; $\alpha = 1$ normal conductivity; $1 < \alpha \le 2$ strong conductivity. Ghazizadeh et al. [56] estimated the fractional order parameter basing on the heat conduction experiment carried out on processed meat by Mitra et al. [57] and observed that $0 < \alpha < 1$ for meat. So we take $0 < \alpha \le 1$ in the present work.

Combining Eq. (1) with the law of conservation of energy, the time-fractional order bioheat conduction equation can be expressed as:

$$\frac{\partial^a}{\partial t^a}\left(\rho c\frac{\partial\theta}{\partial t} + \rho_b w_b c_b (T - T_b) - Q_{met}\right) = k_{ij}^*\theta_{,ji} \quad (0 \le \alpha \le 1) \tag{3}$$

The item $\rho_b w_b c_b (T - T_b)$ in the left side of Eq. (3) describes the heat conduction between blood and tissue. It is assumed $T_b = T_0$ in present work. Thus Eq. (3) can be rewritten as:

$$\frac{\partial^a}{\partial t^a}\left(\rho c\frac{\partial\theta}{\partial t} + \rho_b w_b c_b \theta - Q_{met}\right) = k_{ij}^*\theta_{,ji} \tag{4}$$

An interesting phenomenon can be found in Eq. (4), noting that the equation written in the time-fractional derivative form ($\alpha \ne 1$) is not dimensionally consistent (see Fig. 1), but no one has ever proposed or discussed. Now we will modify this problem.

Analogy to Ferras [58] introduced a parameter to modify the fractional order Pennes' bioheat equation [59]. If a new parameter τ[s] is also introduced, Eq. (4) has the following form: