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A multi-resolution collocation procedure for time-dependent inverse heat problems

Siraj-ul-Islam[a,*], Muhammad Ahsan[a,b], Italf Hussian[a]

[a] Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan
[b] Department of Mathematics, University of Swabi, Pakistan

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ABSTRACT

In this paper, a Haar wavelet collocation method (HWCM) is developed for PDEs related to the framework of so-called inverse problem. These include PDEs with unknown time dependent heat source and unknown solution in interior of the domain. To this end, a transformation is used to eliminate the unknown heat source to obtain a PDE without a heat source. After elimination of unknown non-homogeneous term, an implicit finite-difference approximations is used to approximate the time derivative and Haar wavelets are used for approximation of the space derivatives. Several numerical experiments related to one- and two-dimensional heat sources are included to validate small condition number of coefficient matrix, accuracy and simple applicability of the proposed approach.

1. Introduction

Time-dependent inverse heat problems (IHPs) are among the challenging problems to be solved numerically. IHPs related to PDEs are important due to numerous applications in science and engineering. Such models are encountered in the mathematical modeling of aerospace engineering, nuclear physics, metallurgy, non-destructive testing in stress and strain analysis. Heat conduction problems, optics, communication theory, oceanography, computer vision, cardiography and medical imaging. Such models are in the focus of researchers due to embedded multiple challenges in their numerical solution.

The general form of IHPs to be considered in this paper is:

$$u_t = f(u_{xx}, u_{yy}, u_x, u_y, \mathrm{H}(t)), \quad a < x, y < b, \quad t > 0, \tag{1}$$

where u is any physical phenomena, H(t) is illuminated as either a heat or material source and f represents some physical law. The type of IHP (1) is accompanied by some appropriate initial and boundary conditions.

Since u and the source term H(t) are unknowns in the present case, therefore, due to these unknown terms, existence, uniqueness and stability of the solution is often not assured. Such problems are usually ill-posed [1,2] (as solution does not depend continuously on the boundary and initial conditions) and the numerical results are sensitive to noise in the input data. A number of solution procedures are available in the literature for numerical solution of such inverse problems. In addition to the challenge of ill-posedness, ill-conditioned system matrix of the discretized linear system is another bottleneck encountered in implementation of numerical solution of such type of IHPs.

Various numerical procedures have been reported in the literature [3,4] for numerical solution of IHPs. Implicit schemes are mostly used for the numerical solution of IHPs, whereas the explicit schemes are not much effective [5]. Method of fundamental solution for numerical solution of IHPs are given in Refs. [6–8]. Some relevant numerical methods which are focused on numerical solution of IHPs include boundary element method (BEM) [9], iterative BEM [10,11], Tikhonov regularization technique (TRT) [12,13], operator-splitting method [14], lattice-free high-order finite-difference method [15], third-order mixed-derivative regularization technique [16], Fourier regularization method [17], three-spectral regularization methods [18], and radial basis functions collocation method [19]. A recent work on time-dependent inverse heat problems is given in Ref. [20]. Mallat has also included the inverse problem as a separate chapter in his book [21].

During the last few years, the role of Haar wavelets in numerical computing came to prominence. Researchers have recently used several wavelets techniques for numerical approximation of differential and integral equations. These include, wavelet-based method [22], wavelet meshless methods [23], wavelet collocation methods [24–27], wavelet Galerkin method [28] etc. A view of some of the previous contributions can be found in Ref. [29] and the references there in. Advantages of Haar wavelets for the numerical approximation of different types of problems have been discussed in references [27,30–44,44,45]. Haar wavelets have also been used in other areas like delamination

* Corresponding author.
E-mail address: siraj-ul-islam@uetpeshawar.edu.pk (Siraj-ul-Islam).

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identification [46], magnetic resonance imaging [47], image compression [48], image processing [49], dose calculation [50], detecting and localizing texture defects [51] and in signal processing [52].

Keeping in view the challenges faced in numerical solution of IHPs, a simple and accurate numerical approach using Haar wavelets is proposed in the current paper. The proposed approach produces a stable numerical solution. Unlike the other IHPs specific numerical techniques, the system matrix of the proposed method is comparatively well-conditioned, which has pronounced effect on accuracy of the method. Another advantage of the proposed approach is that different types of boundary conditions can automatically be embedded in the algorithm.

2. Haar wavelets

A Haar wavelet family for $x \in [a, b)$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\zeta_1, \zeta_2), \\ -1 & \text{for} \quad x \in [\zeta_2, \zeta_3), \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

where

$$\zeta_1 = a + (b-a)\frac{k}{m}, \quad \zeta_2 = a + (b-a)\frac{k+0.5}{m}, \quad \zeta_3 = a + (b-a)\frac{k+1}{m}.$$

In the above definition integer $m = 2^j$, $j = 0,1, \ldots, J$, represents the level of the wavelet and integer $k = 0,1, \ldots, m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (2) is calculated using the formula $i = m + k + 1$. In case of minimal values $m = 1$, $k = 0$, we have $i = 2$. The maximal value of i is $2M = 2^{J+1}$. We define the following notations for integrals of the Haar wavelets;

$$P_{i,1}(x) = \int_a^x h_i(x')dx'$$

$$P_{i,2}(x) = \int_a^x P_{i,1}(x')dx'$$

and

$$C_i^H = \int_a^b P_{i,1}(x')dx'.$$

Using Eq. (2), we get

$$P_{i,1}(x) = \begin{cases} x - \zeta_1 & \text{for} \quad x \in [\zeta_1, \zeta_2), \\ \zeta_3 - x & \text{for} \quad x \in [\zeta_2, \zeta_3), \\ 0 & \text{elsewhere}, \end{cases}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\zeta_1)^2 & \text{for} \quad x \in [\zeta_1, \zeta_2), \\ \frac{(b-a)^2}{4m^2} - \frac{1}{2}(\zeta_3 - x)^2 & \text{for} \quad x \in [\zeta_2, \zeta_3), \\ \frac{(b-a)^2}{4m^2} & \text{for} \quad x \in [\zeta_3, 1), \\ 0 & \text{elsewhere}, \end{cases}$$

and

$$C_i^H = \frac{(b-a)^2}{4m^2}.$$

3. Haar wavelets scheme for one-dimensional IHP

Consider the following type of one-dimensional IHP [20,53–56]:

$$u_t(x, t) = u_{xx}(x, t) + \mathrm{H}(t), \quad a \le x \le b, \quad t > 0 \tag{3}$$

with initial and boundary conditions:

$$u(x, 0) = g(x),$$
$$u(a, t) = u_o(t),$$
$$u_x(a, t) = u_l(t),$$

where u and H(t) are the unknown solution and the unknown heat source respectively. For the identification of these unknowns, extra information is provided as $u_x(b, t) = q_o(t)$. In order to solve the IHP numerically, we take the partial derivative of Eq. (3) w.r.t x and use the transformation $u_x(x, t) = v(x, t)$ to get

$$v_t(x, t) = v_{xx}(x, t), \quad a \le x \le b, \quad t > 0. \tag{4}$$

The transformed boundary conditions are

$$v(x, 0) = g'(x),$$
$$v(a, t) = u_l(t),$$
$$v(b, t) = q_o(t).$$

To construct Haar wavelet approximation scheme, we start with the second derivative approximation as

$$v_{xx}(x, t) = \sum_{i=1}^{2M} c_i h_i(x). \tag{5}$$

Integrating Eq. (5) w.r.t x, from a to x and then from a to b, we get

$$v_x(x, t) = v_x(a, t) + \sum_{i=1}^{2M} c_i P_{i,1}(x). \tag{6}$$

$$v_x(a, t) = \frac{v(b, t) - v(a, t)}{b-a} - \sum_{i=1}^{2M} c_i \frac{C_i^H}{b-a}. \tag{7}$$

Putting Eq. (7) in Eq. (6) we get

$$v_x(x, t) = \frac{v(b, t) - v(a, t)}{b-a} + \sum_{i=1}^{2M} c_i \left(P_{i,1}(x) - \frac{C_i^H}{b-a}\right). \tag{8}$$

Again integrating Eq. (8) w.r.t x, from a to x, we get

$$v(x, t) = v(a, t) + (x-a)\frac{v(b, t) - v(a, t)}{b-a} + \sum_{i=1}^{2M} c_i \left(P_{i,2}(x) - (x-a)\frac{C_i^H}{b-a}\right). \tag{9}$$

Let t_o be the current time level and $t = t_o + \Delta t$ be the next time level. By applying the implicit scheme to the time derivative in Eq. (4) we get

$$\frac{v(x, t) - v(x, t_o)}{\Delta t} = v_{xx}(x, t).$$

By re-arranging we get

$$v(x, t) - \Delta t v_{xx}(x, t) = v(x, t_o). \tag{10}$$

Substituting Eq. (9) and Eq. (5) into the Eq. (10) and subsequent discretization at the collocation points $x_k = a + (b-a)\frac{(k-0.5)}{2M}$, $k = 1,2, \ldots, 2M$ leads to the following system of algebraic equations:

$$\sum_{i=1}^{2M} c_i \left[P_{i,2}(x_k) - (x_k - a)\frac{C_i^H}{b-a} - \Delta t h_i(x_k)\right] = v(x_k, t_o) - \left(v(a, t_o) + (x_k - a)\frac{v(b, t_o) - v(a, t_o)}{b-a}\right). \tag{11}$$

Eq. (11) can be written in matrix form as:

$$\left[\mathbf{P}_2 - (\mathbf{x} - a)\frac{\mathbf{C}^T}{b-a} - \Delta t \mathbf{H}\right]\mathbf{c} = \left[v(\mathbf{x}, t_o) - \left(v(a, t_o) + (\mathbf{x} - a)\frac{v(b, t_o) - v(a, t_o)}{b-a}\right)\right] \tag{12}$$

where

$$\mathbf{x} = [x_1, x_2, \ldots, x_{2M}]^T,$$
$$\mathbf{C} = [C_1^H, C_2^H, \ldots, C_{2M}^H]^T,$$
$$\mathbf{c} = [c_1, c_2, \ldots, c_{2M}]^T,$$