International Journal of Thermal Sciences 65 (2013) 148–157

Contents lists available at SciVerse ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

On the solution of the heat equation in very thin tapes

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ARTICLE INFO

Article history:
Received 14 October 2011
Received in revised form
7 September 2012
Accepted 9 October 2012
Available online 7 December 2012

Keywords:
Heat equation
Model reduction
Proper generalized decomposition
Composites manufacturing processes

ABSTRACT

This paper addresses two issues usually encountered when simulating thermal processes in forming processes involving tape-type geometries, as is the case of tape or tow placement, surface treatments, ⋯ The first issue concerns the necessity of solving the transient model a huge number of times because the thermal loads are moving very fast on the surface of the part and the thermal model is usually non-linear. The second issue concerns the degenerate geometry that we consider in which the thickness is usually much lower than the in-plane characteristic length. The solution of such 3D models involving fine meshes in all the directions becomes rapidly intractable despite the huge recent progresses in computer sciences. In this paper we propose to consider a fully space-time separated representation of the unknown field. This choice allows circumventing both issues allowing the solution of extremely fine models very fast, sometimes in real time.

1. Introduction

Industrial processes generally need efficient numerical simulations in order to optimize the process parameters. In the case of composite materials, even if the thermo-mechanical models are nowadays well established, efficient simulations need for further developments.

In this work we are considering some issues, analyzed from a methodological point of view, without considering its industrial counterpart that requires the coupling of different numerical procedures and richer physics.

Thermal models involved in the numerical modeling of composite tape placement processes introduce, despite its geometrical simplicity, a certain number of numerical difficulties related to: (i) the very fine mesh required due to the small domain thickness with respect to the other characteristic dimensions as well as to the presence of a thermal source moving on the domain surface; and (ii) the long simulation times induced by the low thermal conductivity of polymers and the movement of the heat source;

The solution by using standard discretization techniques can be extremely expensive from the computing time point of view. For example, if one wants to simulate a thermal problem in a ply whose thickness is 1000 times lower than its length (which is a quite common ratio), the use of only 100 nodes in the thickness will lead to use 10^5 nodes in the length to ensure the geometrical quality of the mesh on which standard discretization techniques, like the finite element method, proceed. The total amount of nodes is then 10 millions even when considering a 2D thermal model. In this situation solving a 3D model seems a challenge. Indeed, when the model involves 10^{12} (that implies a reasonable number of nodes, of the order of 10^4 in each coordinate direction of a 3D model) numerical complexity reaches the current computer capabilities. In addition, in transient non-linear models the problem must be solved at least once at each time step, time step that can be extremely small due to stability constraints.

In order to reduce the computing time needed for solving large numerical models, different ways have been explored. One consists in using super high performance computing facilities. Others strategies consider subdomains, multigrid techniques or the use of efficient preconditioners. Another efficient way to enhance the simulation capabilities is to reduce the size of the approximation basis employed for approximating the unknown field. In the finite elements method, at least one approximation function is associated to each node. Thus, the number of degrees of freedom scales with the number of nodes. Reduced modeling lies in using a reduced number of "appropriate" approximation functions defined in general in the whole domain and able to approximate up to a certain level of accuracy the problem solution at each time. Thus, the number of approximation functions (and by the way the

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http://dx.doi.org/10.1016/j.ijthermalsci.2012.10.017

number of degrees of freedom) becomes independent of the mesh size. The arising issue is how to calculate these "appropriate" functions defining the reduced approximation basis?

There are several possibilities. A first possibility lies in the use of the Proper Orthogonal Decomposition – POD – that was employed in a former work [9] for addressing similar issues to the ones concerned by the present work. In what follows we are describing how the POD extracts relevant information for building-up a reduced approximation basis.

1.1. Extracting relevant information by applying the proper orthogonal decomposition

We assume that the field of interest $u(\mathbf{x},t)$ is known at the nodes $\mathbf{x}_i$ of a spatial mesh for discrete times $t_m = m\cdot\Delta t$, with $i\in[1,\cdots,M]$ and $m\in[0,\cdots,P]$. We use the notation $u(\mathbf{x}_i,t_m)\equiv u^m(\mathbf{x}_i)\equiv u_i^m$ and define $\mathbf{u}^m$ as the vector of nodal values u_i^m at time t_m. The main objective of the POD is to obtain the most typical or characteristic structure $X(\mathbf{x})$ among these $u^m(\mathbf{x})$, $\forall m$. For this purpose, we solve the following eigenvalue problem [32]:

$$\mathbf{C}X = \alpha\mathbf{X}. \tag{1}$$

Here, the components of vector $\mathbf{X}$ are $X(\mathbf{x}_i)$, and $\mathbf{C}$ is the two-point correlation matrix

$$\mathbf{C}_{ij} = \sum_{m=1}^{P} u^m(\mathbf{x}_i)\cdot u^m(\mathbf{x}_j), \tag{2}$$

whose matrix form reads:

$$\mathbf{C} = \sum_{m=1}^{P} \mathbf{u}^m\cdot(\mathbf{u}^m)^T, \tag{3}$$

which is symmetric and positive definite. With the matrix $\mathbf{Q}$ defined as

$$\mathbf{Q} = \left(\mathbf{u}^1,\cdots,\mathbf{u}^P\right) \tag{4}$$

We have

$$\mathbf{C} = \mathbf{Q}\cdot\mathbf{Q}^T. \tag{5}$$

1.2. Building the POD reduced-order model

In order to obtain a reduced model, we first solve the eigenvalue problem Eq. (1) and select the N eigenvectors $\mathbf{X}_i$, $i = 1,\cdots,N$, associated with the N eigenvalues belonging to the interval defined by the highest eigenvalue α_1 and α_1 divided by a large enough number (e.g. 10^8). In practice, N is found to be much lower than M. These N eigenfunctions $\mathbf{X}_i$ are then used to approximate the solution $u^m(\mathbf{x})$, $\forall m$. To this end, let us define the matrix $\mathbf{B} = (\mathbf{X}_1\cdots\mathbf{X}_N)$.

Now, let us assume for illustrative purposes that an explicit time-stepping scheme is used to compute the discrete solution $\mathbf{u}^{m+1}$ at time t^{m+1}. One must thus solve a linear algebraic system of the form

$$\mathbf{G}^m\mathbf{u}^{m+1} = \mathbf{H}^m. \tag{6}$$

A reduced-order model is then obtained by approximating $\mathbf{u}^{m+1}$ in the subspace defined by the N eigenvectors $\mathbf{X}_i$, i.e.

$$\mathbf{u}^{m+1}\approx\sum_{i=1}^{N}\mathbf{X}_i\cdot T_i^{m+1} = \mathbf{B}\cdot\mathbf{T}^{m+1}. \tag{7}$$

Eq. (6) then reads

$$\mathbf{G}^m\cdot\mathbf{B}\cdot\mathbf{T}^{m+1} = \mathbf{H}^m, \tag{8}$$

or equivalently

$$\mathbf{B}^T\cdot\mathbf{G}^m\cdot\mathbf{B}\cdot\mathbf{T}^{m+1} = \mathbf{B}^T\cdot\mathbf{H}^m. \tag{9}$$

The coefficients $\mathbf{T}^{m+1}$ defining the solution of the reduced-order model at the time step $m+1$ are thus obtained by solving an algebraic system of size N instead of M. When $N \ll M$, as is the case in numerous applications, the solution of Eq. (9) is thus preferred because of its much reduced size.

Remark 1 The reduced-order model Eq. (9) is built a posteriori by means of the already-computed discrete field evolution. Thus, one could wonder about the interest of the whole exercise. In fact, two beneficial approaches are widely considered (see e.g. [6,8,18,24–26,31,32]). The first approach consists in solving the large original model over a short time interval, thus allowing for the extraction of the characteristic structure that defines the reduced model. The latter is then solved over larger time intervals, with the associated computing time savings. The other approach consists in solving the original model over the entire time interval, and then using the corresponding reduced model to solve very efficiently similar problems with, for example, slight variations in material parameters or boundary conditions. We considered some years ago an adaptive technique for constructing the reduced basis without an "a priori" knowledge [2,31,32], following the original proposal in [30].

Remark 2 The construction of the reduced bases is not unique. There are many alternatives. Some ones introduce some improvements on the POD methodology just described, as is the case of the Goal Oriented Model Constrained Optimization approach (see [7] and the references therein) or the modal identification method (see [13] and the references therein). The Branch Eigenmodes Reduction Method combined with the amalgam method is another appealing constructor of reduced bases [34].

Remark 3 The application of the POD allows to express the unknown function $u(\mathbf{x},t)$ in the reduced space-time separated form

$$u(\mathbf{x},t)\approx\sum_{i=1}^{i=N}T_i(t)\cdot X_i(\mathbf{x}) \tag{10}$$

where $X_i(\mathbf{x})$ are space dependent function (the eigenfunctions resulting from the application of the POD) and $T_i(t)$ are its coefficients that only depend on time.

1.3. From POD to PGD

Despite the fact of having proposed techniques able to define the reduced basis without an "a priori" knowledge, the robustness of such strategies is not ensured and in some cases these strategies do not converge. In that case one could consider as starting point a separated representation of the problem solution $u(\mathbf{x},t)$

$$u(\mathbf{x},t)\approx\sum_{i=1}^{i=N}T_i(t)\cdot X_i(\mathbf{x}) \tag{11}$$

and then inject it in the weak form of the problem. This procedure allows computing the functions involved in the separated approximation without any "a priori" knowledge. This strategy was proposed by Pierre Ladeveze in the 80's, and he called it radial approximation [19,20,23].

Inspired by this procedure one could try to generalize this representation to the multidimensional fields as was proposed