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Laminar shallow viscoplastic fluid flowing through an array of vertical obstacles

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ABSTRACT

A new Bingham-Darcy shallow depth approximation flow model is proposed in this paper. This model is suitable for a shallow viscoplastic fluid flowing on a general topography and crossing an array of vertical obstacles. An analogous porous medium is first introduced to reduce the array of obstacles. The reduction model is based on a continuum model similar to the Brinkman equations, where the usual Darcy model is extended for viscoplastic Bingham fluids. A specific asymptotic analysis of this Bingham-Darcy porous medium for the case of shallow depth flows allows us to produce a new reduced model. Some assumptions are needed for the reduction: laminar flow, small degrees of slope variation of the underlying topography and a yield stress that is small when compared with gravity effects. The resulting solution is a highly nonlinear parabolic equation in terms of the flow height only, and is efficiently solved by a Newton method, without any regularization. However, our numerical predictions compares well, both qualitatively and quantitatively with both experimental measurements and full tridimensional simulations. Finally, a new experiment for a viscoplastic flow over an inclined plane through a network of obstacles is proposed and numerical simulations are provided for future comparison with experiments.

1. Introduction

The problem of complex fluids flowing through networks of discrete obstacles applies to many applications in natural and material sciences. During natural risk assessments, for example, volcanic debris and/or lava flows may move through dense forests, as was the case for lavas advancing during Kilauea's July 1974 eruption [1] and Etna's 2002-03 eruption [2], among others. To date, lava flow emplacement models have tended to consider tree-free surfaces in completing their simulations (e.g., [3–5]). The same applies to non-volcanic debris flows in forested mountainous or urban areas (see e.g. [6,7]). In terms of material sciences, flow of a viscoplastic fluid through arrays of solid cylinders needs to be considered in industrial processes, such as in the case of fresh concrete spreading through networks of steel bars (see e.g. [8,9]).

Taking into account each obstacle in numerical simulations leads to very time consuming computations. The usual approach is to replace the discrete configuration of obstacles with an equivalent continuous medium, the so-called fibrous porous medium. In the case of a Newtonian fluid, this continuous medium is described by the classic Darcy model [10]. This model proposes a linear relation between the flow rate and the pressure drop occurring across the porous medium. In 1949, Brinkman [11] proposed a modification of the classic Darcy model by combining the Navier–Stokes equations with the Darcy model. This combination is useful for situations where there are both flow sub-regions with and without porous media. While the Brinkman model provides a global description of both these sub-regions, the Darcy model alone is unable to describe regions without porous media. Conversely, in the case of a non-Newtonian fluid, the situation is rather complex. This is due to , from one hand, the complexity of the fluid behavior and, on the other hand, the porous micro-structure. Bourgeat and Mikelić [12] proposed a first theoretical analysis of quasi-Newtonian shear-thinning and shear-thickening fluids, and derived a modified-Darcy model. This modified-Darcy model involves an effective viscosity (η_{eff}), which depends both upon the flow rate and the micro-structural characteristics of the porous medium, these being described by the permeability tensor ($\boldsymbol{\kappa}$) and porosity (ϕ). The porosity is defined as the volume fraction of fluid in the mixture represented by fluid as opposed to the obstacles: it is equal to one for a medium without obstacles and tends to zero as the obstacle density increases. Permeability depends both upon ϕ and the geometric configuration of the obstacles, where some explicit expressions of $\boldsymbol{\kappa}$ vs ϕ exist, and these depend on a

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geometrical hypothesis based upon the obstacle distribution. For Non-Newtonian viscoplastic fluids, a yield stress has to be reached to achieve a transition between the unyielded, arrested state of the fluid and the yielding state (see e.g. [13,14]). Thus, there fluids which do not obey to the usual linear Darcy model with their rheological response to an applied stress, as there is also some no-flow situations in porosities [15,16]. Based on an experimental investigation, Pascal [17] proposed – for an Herschel–Bulkley viscoplastic fluid – a modified-Darcy model that uses a threshold gradient. Some papers have since focused on flow through packed beds of spherical particles. Al-Fariss and Pinder [18], for example extended the Pascal [17] model by deriving an equation for the threshold gradient to describe the flow of waxy oil through beds of packed spheres. Chevalier et al. [19], based on experimental measurements for a fluid with a yield stress passing through packed glass beads, proposed an empirical relationship between the pressure drop and the flow rate. Recent studies have considered complex fluid flow through fibrous porous media. Bleyer and Coussot [20], for example performed two-dimensional numerical simulations of a viscoplastic flow through an ordered array of disks; Shahsavari and McKinley [21] investigated the flow of Herschel-–Bulkley fluids through fibrous media by means of numerical studies and scaling analyse. As a result, Shahsavari and McKinley [21] developed an effective viscosity function which can be used in the modified-Darcy model for steady fully-developed viscoplastic flows. In addition, Vasilic et al. [8] defined the apparent shear rate using a shift factor and a generalized Brinkman equation. They also performed numerical simulations using a bi-viscosity regularized viscoplastic model and compared their results with experimental measurements on a Carbopol gel flow which was slowly poured into a transparent container where an array of cylindrical steel were located across the middle zone of simulation. Comparing simulations and experimental observations, they obtained both qualitative and quantitative agreements for the final shape of the flow.

For thin flows, the usual approach is to consider shallow-depth approximations. The shallow-flow approximations of laminar viscoplastic Bingham fluids were first studied by Liu and Mei [22], based on a rigorous asymptotic analysis. This approach was revisited by Balmforth and Craster [23] and extended to the axisymmetric case [24], for an application to volcanic lava domes. For fast flows, such as debris and mud flows on mountain slopes, Laigle and Coussot [25] derived the first reduced model that combines both inertial and viscoplastic effects, where viscoplastic effects are estimated from the friction at the flow base. Assuming a compressible material, Bresch et al. [26] derived a reduced viscoplastic model that also included inertial effects. This approach was next revisited in the incompressible case in terms of asymptotic analysis by Fernández-Nieto et al. [27] and by Ionescu [28] who both applied an augmented Lagrangian algorithm. Practical predictions of natural hazard need to take into account general tridimensional and complex topographies (see e.g. [29]). A new approach for topography in shallow flow models was proposed by Bouchut et al. [30] which relaxes most restrictions, such as slowly varying topographies. Next, Ionescu [31], considered Bingham and Drucker–Prager models and extended this approach with an elegant formulation based on surface differential operators (surface gradient and divergence) while also including inertia effects. For a more exhaustive review of various shallow flow approximations of viscoplastic fluids, see the recent review paper of Saramito and Wachs [13].

The model proposed here allows a shallow-flow approximation of both the Bingham-Brinkman model, and involved application a modified-Darcy model for viscoplastic fluids. This model is of practical application for risk assessment, by opening the possibility of numerically investigating the effects of forests on the spatial and temporal of volcanic flow propagation, which has an unknown effect on flow advance (see [32]). Our model could be also useful for industrial processes, such as fresh concrete spreading through arrays of steel bars, as the required computing time is dramatically decreased over fully three-dimensional models. Instead of time-dependent tridimensional simulations with moving free-surfaces, the model presented here requires only the solution of a simple two-dimensional parabolic equation for flow height. The array of obstacles is first reduced to a continuum model by a generalized tensor Brinkman equations for yield strength fluids. Then, by assuming a shallow laminar flow, a low slope variation in the topography and a yield stress that is assumed to be small when compared to the gravity effects, we extend a previous asymptotic analysis [29] to the Brinkman equations extended for the Bingham model.

The outline of this paper is, thus, as follows. Section 2 proposes a new shallow-depth approximation of the viscoplastic Bingham model for a fluid flowing on a general topography and crossing an array of vertical obstacles. Section 3 proposes a Newton's algorithm for efficient solution of an unregularized nonlinear Bingham-Brinkman (reduced) model. Comparisons between numerical simulations and experimental observations are presented and discussed in Section 4. Finally, the flow of viscoplastic fluids on an inclined plane through different fibrous mediums is numerically investigated, our aim being to understand and quantify the influence of the obstacles on the flow propagation. This numerical experiment could be reproduced with real fluids for future comparisons and benchmarking. The impatient reader, who is not interested in the asymptotic analysis, can read paragraph 2.1 for the initial problem, then paragraph 2.5 for the final reduced problem and, finally, jump straight to Section 4 for a review of our results and a discussion.

2. Bingham-Darcy shallow depth approximation

2.1. The initial tridimensional problem

We consider the Bingham model [33] constitutive equation which expresses the deviatoric part $\boldsymbol{\tau}$ of the stress tensor versus the rate of deformation tensor $\dot{\boldsymbol{\gamma}}$ as:

$$\begin{cases} \tau = \eta\dot{\gamma} + \tau_y \dfrac{\dot{\gamma}}{|\dot{\gamma}|} & \text{when}\,\dot{\gamma} \neq 0, \\ |\tau| \leq \tau_y & \text{otherwise,} \end{cases} \tag{1}$$

where $\eta > 0$ is the plastic viscosity and $\tau_y \geq 0$ is the yield stress. Here $|\tau| = \left((1/2)\sum_{i,j=1}^{3} \tau_{ij}^2\right)^{1/2}$ denotes the conventional norm of a symmetric tensor in mechanics. The total Cauchy stress tensor is $\boldsymbol{\sigma} = -p.\,\mathbf{I} + \boldsymbol{\tau}$ where p is the pressure and $\mathbf{I}$ the identity tensor. We assume that the array of obstacles can be treated as an equivalent continuum porous medium. The constitutive Eq. (1) is then completed by the conservations of momentum and mass:

$$\rho(\partial_t \mathbf{u} + (\mathbf{u}.\,\nabla)\mathbf{u}) - \mathbf{div}(\tau) + \nabla\mathbf{p} \;=\; \mathbf{f_p} + \rho\mathbf{g}, \tag{2}$$

$$\mathrm{div}\,\mathbf{u} \;=\; 0, \tag{3}$$

where $\rho > 0$ is the constant density, $\mathbf{g}$ is the gravity vector and $\mathbf{f}_p$ a source term based on local generalized Darcy's law (e.g. see [34]) relating the force exerted on the pore fluid (typically gradient pressure and gravity force) to the macroscopic-scale velocity by:

$$\mathbf{f_p} = -\eta_{\text{eff}}\boldsymbol{\kappa}^{-1}\mathbf{u}, \tag{4}$$

where $\eta_{\text{eff}} \geq 0$ is the material local apparent viscosity and $\boldsymbol{\kappa}$ the permeability tensor. The conservation equations with the addition of a Darcy source term in the momentum equation is called the Brinkman equations [11]. This model allows us to deal with mixed cases where only a part of the calculation domain is taken up by a fibrous porous medium. In this case, out side of the porous zone, the permeability is infinite and the source term $\mathbf{f}_p$ vanishes to give the standard conservation equations.

It was proved in [35,36] that the permeability tensor is symmetric and positive definite. This means that the permeability tensor has three