Journal of Non-Newtonian Fluid Mechanics 000 (2015) 1–8

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: www.elsevier.com/locate/jnnfm

Development of shear bands for a model of a thixotropic yield stress fluid

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ARTICLE INFO

Article history:
Received 2 June 2015
Revised 14 October 2015
Accepted 16 October 2015
Available online xxx

Keywords:
Shear banding
PEC model
Yield stress
Asymptotic methods
Wormlike micelles

ABSTRACT

The PEC (partially extending strand convection) model of Larson is able to describe thixotropic yield stress behavior in the limit where the relaxation time is large relative to the retardation time. In this paper, we discuss the development of shear bands in a Poiseuille flow which is started up from rest with an imposed pressure gradient. We analyze the asymptotic limit of large relaxation time; the small parameter ϵ measures the ratio of retardation time to relaxation time. We determine the position and width of shear bands as a function of time. We identify an initial phase of “fast yielding” during which the width of the transition between high and low shear rate regions behaves like t^{-3}. This continues until t (measured on the scale of the retardation time) is on the order of $\epsilon^{-1/3}$. Then there is a phase of “delayed yielding” during which the width of the transition is of order ϵ. Eventually, the width sharpens as $1/(\epsilon^2 t^3)$. We also show how these results are modified by introducing Korteweg stresses which prevent the transition from becoming infinitely sharp and also change the location where the transition takes place.
This paper is dedicated to Roger Tanner on the occasion of his eighty-second birthday.

1. Introduction

Experiments on many yield stress fluids show phenomena which are not explained by traditional models like Bingham or Herschel–Bulkley, such as stress overshoots, yield stress hysteresis, shear banding, “delayed yielding” (i.e. the fluid yields at a lower stress when the stress is imposed over a longer time), and thixotropy. We refer to, for instance [4–7,14,19,22,28].

Theoretical efforts to capture complex yield behavior often add a structure parameter to “simple” yield stress fluids such as the Bingham or Herschel–Bulkley model. For instance, the viscosity [24] or the power law exponent [11] may depend on the structure parameter, which in turn is influenced by the deformation. An alternative approach is a viscoelastic model, in which yield stress behavior arises in the limit where the relaxation time becomes large. Such an approach can be viewed as a precise mathematical elaboration of the thesis that the yield stress is a “myth” and that apparent yield stress behavior hides phenomena on unresolved time scales [3]. In [25], the behavior of the PEC (partially extending strand convection) model of Larson [16] in homogeneous shear flow was analyzed for the limit of large relaxation time and was found to exhibit the salient features of thixotropic yield stress fluids. This model was first proposed for entangled polymer melts, but has recently been applied to wormlike micelles [31]. Such fluids are often viewed as “apparent” yield stress fluids, since they undergo a yielding transition where the viscosity jumps by two or three orders of magnitude, but they can be observed to flow in the unyielded state.

A separation of time scales is essential for thixotropic behavior. Basically, such behavior depends on a slow evolution of microstructure in apparent absence of deformation. The limit of large relaxation time provides such a separation of time scales. In the PEC model, the yield stress behavior arises naturally as a consequence of the model rather than being built into it. Singularly perturbed dynamical systems with “fast” and “slow” time scales arise in the analysis, leading to three asymptotic regimes, a fast elastic regime, a flowing yielded regime, and a slow dynamics during which there is almost no deformation, but the microstructure slowly changes.

We follow the formulation of the PEC model given in [25]; the equivalence to Larson’s original formulation is discussed there. The model can be described in terms of a conformation tensor **C**, which satisfies a differential equation of the form

$$\mathbf{C}^{\nabla} + \epsilon(\phi(\operatorname{tr}\mathbf{C})\mathbf{C} - \chi(\operatorname{tr}\mathbf{C})\mathbf{I}) = 0. \quad (1)$$

Here $\mathbf{C}^{\nabla}$ denotes the upper convected time derivative. The relaxation time $1/\epsilon$ is assumed to be large. The stress tensor **T** is related to **C** by

$$\mathbf{T} = \psi(\operatorname{tr}\mathbf{C})\mathbf{C}. \quad (2)$$

In this formulation, ϕ, ψ and χ are constitutive functions, which can in principle be specified arbitrarily. We can view this model as a modification of the upper convected Maxwell model, in which the viscosity and relaxation time depend on the “structure parameter” $\operatorname{tr}\mathbf{C}$. With this interpretation, the model has a certain similarity with other

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http://dx.doi.org/10.1016/j.jnnfm.2015.10.004

efforts to model yield stress fluids; the main difference being that the starting point is a viscoelastic theory rather than a yield stress theory. The model in the form given above is similar to dumbbell based models of dilute polymer solutions. However, the constitutive functions are quite different from those typically used in that situation. For the PEC model, we specifically have, with s denoting $\mathrm{tr}\,\mathbf{C}$, $\phi(s) = \chi(s) = s + \alpha$, $\psi(s) = k/(s + \alpha)$, where k is a stress modulus and $\alpha > -3$ is a dimensionless constant.

In parallel shear flow, we have $C_{22} = C_{33} = 1$, $C_{13} = C_{23} = 0$, $s = C_{11} + 2 + \alpha$, and the above model reduces to

$$\dot{C_{11}} = 2\kappa C_{12} + \epsilon(s+\alpha)(1 - C_{11}),$$
$$\dot{C_{12}} = \kappa - \epsilon(s+\alpha)C_{12}. \tag{3}$$

Here κ is the shear rate, the flow is in the "1" direction, and the velocity varies across the "2" direction. We assume that the total stress consists of the stress given by the PEC model plus a Newtonian stress with viscosity η. Hence if τ is the total shear stress, we have

$$\kappa := \frac{1}{\eta}(\tau - T_{12}) = \frac{1}{\eta}\left(\tau - \frac{kC_{12}}{s+\alpha}\right). \tag{4}$$

The parameter k has the dimension of stress; specifically, the linear stress modulus is equal to $k/(3 + \alpha)$. There are two relevant time scales in this model; the relaxation time

$$\tau_1 = \frac{1}{(3+\alpha)\epsilon},$$

and the retardation time

$$\tau_2 = \frac{(3+\alpha)\eta}{k}.$$

Throughout, our analysis will be based on assuming that τ_1/τ_2 is large. We can alternatively characterize τ_1/τ_2 as the ratio of unyielded to yielded viscosity.

We shall nondimensionalize stresses by scaling with k, so we shall set $k = 1$ in the following. Moreover, we shall scale time with η/k, so we shall also set $\eta = 1$. The parameter ϵ is then a dimensionless measure of an inverse relaxation time. We can also eliminate α from the equations. Specifically, if we scale C_{12} with $\sqrt{3+\alpha}$, $C_{11} - 1$ with $3 + \alpha$, τ with $1/\sqrt{3+\alpha}$, time with $3 + \alpha$ and ϵ with $(3 + \alpha)^{-2}$, then α scales out of the equations. For the rest of this paper, we have arbitrarily set $\alpha = 1$.

The steady shear behavior of the PEC model shows a nonmonotone stress versus shear rate curve. There is therefore a range of shear stresses where two different shear rates (a "yielded" and "unyielded" state) are possible and a range of shear rates where no stable steady state exists. This leads to the phenomena of shear banding and shear stress hysteresis, see for instance [20,23]. In this paper, we shall be concerned with the time-dependent startup of shear flow under a given imposed shear stress.

The asymptotic analysis of solutions to (3) involves the interplay of several dynamic regimes. The simplest of these is obtained simply be setting $\epsilon = 0$ in the equations. We shall refer to the resulting limit as "fast" dynamics. There are two potential reasons why the neglected ϵ terms may become relevant, however:

1. The observation time is long (of order $1/\epsilon$), and therefore, even though the short term effect of the ϵ term is negligible, its cumulative effect must be considered. This will be referred to as "slow" dynamics.
2. The components of $\mathbf{C}$ are large, so even if ϵ is small, it is multiplied by a large number in the equations. This leads to "yielded" dynamics.

In [25], the startup of shear flow was considered. That is, we start from an initial condition $C_{12} = 0$, $C_{11} = 1$, impose a constant value of the shear stress τ and then follow the evolution of the solution of (3). As shown in [25], the behavior can be described as follows:

1. If $\tau > 1/4$, then fast yielding occurs. The dynamics transitions from fast to yielded dynamics.
2. If $1/(4\sqrt{2}) < \tau < 1/4$, delayed yielding occurs. The dynamics transitions from fast to slow dynamics, but after a long time (of order $1/\epsilon$), there is another transition from slow to fast and eventually to yielded dynamics.
3. If $\tau < 1/(4\sqrt{2})$, slow dynamics reaches a steady state, and the flow remains unyielded.

In this paper, we shall consider the start up of plane Poiseuille flow. The flow is in the x direction, and the velocity varies in the y direction (in the index notation used above, these direction correspond to index 1 and 2, respectively). For negligible inertia, the momentum balance is given by

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}. \tag{5}$$

If Q is the imposed pressure gradient, we therefore have $\partial\tau/\partial y = Q$, and if the centerline of the channel is at $y = 0$, we find $\tau = Qy$. Therefore, Poiseuille flow with imposed pressure gradient is equivalent to a one-parameter family of problems with imposed shear stress; this imposed shear stress varies linearly with location. We shall choose the imposed pressure gradient large enough to cause yielding near the channel walls, while at the center of the channel the shear stress is always zero, so the material remains unyielded. A sharp transition layer forms which separates yielded and unyielded regions. To understand the behavior of this transition region, we need to analyze not just the temporal behavior of solutions for given τ, but also the variation of the solution with τ. We can nondimensionlize length so that $Q = 1$, i.e. $\tau = y$. We therefore do not need to distinguish in the following between the imposed shear stress and the location in the channel. We shall assume that the half width of the channel is at least 1/4, so that fast yielding will occur near the walls of the channel.

We remark that the flow analyzed in this paper is stress controlled, while most experiments are strain controlled. In a strain controlled experiment, thixotropic yield stress fluids will exhibit shear banding with a homogeneous stress (see e.g. [20,23]. On the other hand, in a stress controlled experiment which starts from the unyielded state, the fluid remains unyielded as long as this is possible, and will yield only if the imposed stress exceeds the maximum in the steady flow curve.

2. The development of shear bands

The analysis in the following will focus on the yield time, i.e. the time which passes between the initial start up of the flow and the time when the fluid yields. The yield time will be of order 1 if $\tau > 1/4$, of order $1/\epsilon$ if $1/(4\sqrt{2}) < \tau < 1/4$, and there there is no yielding if $\tau < 1/(4\sqrt{2})$. A more complicated analysis is needed to elucidate the specifics of the transitions between these regimes when τ is close to 1/4 or, respectively, $1/(4\sqrt{2})$. We shall discuss various rescalings of the equations leading to certain asymptotic regimes in these transition zones. Before we analyze this in detail, we discuss a relationship between the variation of the yield time as a function of τ, and the width of the transition between yielded and unyielded regions.

Let $T(\tau, \kappa)$ be the time when the shear rate reaches κ. We can pick κ to be a suitable value and identify the position of the yield zone at time T as the inverse function $\tau(T, \kappa)$. To determine the width of the transition region, let $K(\tau, t)$ be the shear rate at position τ and time t; this shear rate is given by (4) with C_{12}, C_{11} given by the solution of (3). Clearly, we have

$$K(\tau, T(\tau, \kappa)) = \kappa. \tag{6}$$

We differentiate this equation with respect to τ and find

$$K_\tau + K_t T_\tau = 0. \tag{7}$$