Journal of Non-Newtonian Fluid Mechanics 000 (2015) 1–14

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: www.elsevier.com/locate/jnnfm

Cessation of viscoplastic Poiseuille flow in a square duct with wall slip

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ARTICLE INFO

Article history:
Received 21 July 2015
Revised 3 November 2015
Accepted 10 November 2015
Available online xxx

Keywords:
Bingham plastic
Slip
Slip yield stress
Poiseuille flow
Rectangular duct
Papanastasiou regularization
Cessation

ABSTRACT

We solve numerically the cessation of the pressure-driven Poiseuille flow of a Bingham plastic under the assumption that slip occurs along the wall following a generalized Navier-slip law involving a non-zero slip yield stress. In order to avoid the numerical difficulties caused by their inherent discontinuities, both the constitutive and the slip equations are regularized by means of exponential (Papanastasiou-type) regularizations. As with one-dimensional Poiseuille flows, in the case of Navier slip (zero slip yield stress), the fluid slips at all times, the velocity becomes and remains plug before complete cessation, and the theoretical stopping time is infinite. The cessation of the plug flow is calculated analytically. No stagnant regions appear at the corners when Navier slip is applied. In the case of slip with non-zero slip yield stress, the fluid may slip everywhere or partially at the wall only in the initial stages of cessation depending on the initial condition. Slip ceases at a critical time after which the flow decays exponentially and the stopping times are finite in agreement with theory. The combined effects of viscoplasticity and slip are investigated for wide ranges of the Bingham and slip numbers and results showing the evolution of the yielded and unyielded regions are presented. The numerical results also showed that the use of regularized equations may become problematic near complete cessation or when the velocity profile becomes almost plug.

1. Introduction

We have recently solved numerically the steady-state Poiseuille flow of Herschel–Bulkley fluids in a duct of rectangular cross section under the assumption that slip occurs along the wall only if the wall shear stress, τ_w, exceeds a critical value, τ_c, known as the slip yield stress [1]. For this purpose, we employed the following slip equation

$$\begin{cases} u_w = 0, & \tau_w \le \tau_c \\ \tau_w = \tau_c + \beta u_w, & \tau_w > \tau_c \end{cases} \tag{1}$$

where u_w is the slip velocity, defined as the relative velocity of the fluid with respect to that of the wall, and β is the slip parameter, which depends on the temperature, and on the properties of the material and of the fluid/wall interface [2]. A literature review of experimental observations of slip yield stress with both Newtonian and non-Newtonian materials is provided in Ref. [3]. When the slip yield stress vanishes, Eq. (1) is reduced to the well-known Navier slip equation:

$$\tau_w = \beta u_w \tag{2}$$

The no-slip and the perfect-slip cases correspond to $\beta \to \infty$ and $\beta = 0$, respectively. In most experimental studies on various materials τ_c appears to be much lower than the yield stress τ_0 [4–6]. In general, the relative values of τ_c and τ_0 may lead to different flow regimes (see, e.g., Refs. [1,6]). In the present work, τ_c is taken equal to τ_0, as suggested by Pearson and Petrie [7], in order reduce the number of the flow regimes.

In [1], it has been demonstrated that there are four distinct regimes in steady-state Poiseuille flow in a rectangular duct, defined by three critical values of the pressure gradient. Initially no slip occurs, in the second regime slip occurs only in the middle of the wider wall, in the third regime slip occurs partially at both walls, and eventually variable slip occurs everywhere. The two intermediate partial-slip regimes collapse to one in the case of a square duct.

In order to study the combined effects of viscoplasticity and slip in this steady-state flow, Damianou and Georgiou [1] employed the Herschel–Bulkley constitutive equation. In the present work, we solve the time-dependent flow. In order to reduce the number of parameters involved we consider here the flow of a Bingham plastic in a square duct. The Bingham constitutive equation relates the viscous stress tensor $\boldsymbol{\tau}$ to the rate-of-strain tensor $\dot{\boldsymbol{\gamma}}$ as follows

$$\begin{cases} \dot{\boldsymbol{\gamma}} = \mathbf{0}, & \tau \le \tau_0 \\ \boldsymbol{\tau} = \left(\dfrac{\tau_0}{\dot{\gamma}} + \mu\right)\dot{\boldsymbol{\gamma}}, & \tau > \tau_0 \end{cases} \tag{3}$$

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http://dx.doi.org/10.1016/j.jnnfm.2015.11.002
0377-0257/

where μ is the plastic viscosity, and τ and $\dot{\gamma}$ are the magnitudes of $\boldsymbol{\tau}$and $\dot{\boldsymbol{\gamma}}$, defined respectively by

$$\dot{\gamma} \equiv \sqrt{\frac{1}{2} II_{\dot{\gamma}}} = \sqrt{\frac{1}{2} \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}} \quad \text{and} \quad \tau \equiv \sqrt{\frac{1}{2} II_{\tau}} = \sqrt{\frac{1}{2} \boldsymbol{\tau} : \boldsymbol{\tau}} \tag{4}$$

where the symbol II stands for the second invariant of a tensor. Finally, the rate-of-strain tensor is defined by

$$\dot{\boldsymbol{\gamma}} \equiv \nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}} \tag{5}$$

where $\mathbf{u}$ is the velocity vector and the superscript T denotes the transpose.

The two-branch Eq. (3) predicts that the material behaves like a solid in regions where the stress is below the yield stress ($\tau \le \tau_0$) and as a fluid in regions where the yield stress is exceeded ($\tau > \tau_0$). These are called unyielded and yielded regions, respectively. It should be noted that unyielded regions include not only regions where the material is stagnant (dead zones) but also zones where the material moves undeformed as a rigid body. The determination of the unyielded and yielded regions, where the two branches of the constitutive equation apply, is a major issue in solving viscoplastic flows. As emphasized by Huilgol [8], this task becomes more difficult in unsteady Bingham flow, since the position and shape of the yielded and unyielded regions has to be determined as a function of space and time. Using regularized versions of the constitutive equation has been a very popular approach in tackling this problem. The main alternative approach is the use of Augmented Lagrangian Methods (ALMs), which are based on the use of variational inequalities. The advantages and disadvatanges of the two approaches are well known and have been recently reviewed by Glowinski and Wachs [9] and by Balmforth et al. [10].

As in [1], instead of the two-branch Eq. (3) we employ the Papanastasiou regularization [11]:

$$\boldsymbol{\tau} = \left\{ \frac{\tau_0 [1 - \exp(-m\dot{\gamma})]}{\dot{\gamma}} + \mu \right\} \dot{\boldsymbol{\gamma}} \tag{6}$$

where m is the stress growth exponent. The above model applies everywhere in the flow field (in both yielded and practically unyielded regions) and at the same time provides a satisfactory approximation of the Bingham model for sufficiently large values of the parameter m. This has been tested in numerous benchmark problems [11,12].

The analogy between Eqs. (1) and (3) is obvious. Difficulties analogous to those encountered when using the discontinuous Bingham model also arise when employing the discontinuous slip Eq. (1). Together with the unknown velocity and pressure fields, one has to determine the regions of the wall where slip occurs and those where the no-slip boundary condition applies. Such a task may be trivial to deal with in the case of steady one-dimensional Poiseuille flows but it becomes very difficult in the case of time-dependent two- and three-dimensional flows. Even in the case of Newtonian flows, it is not possible to obtain analytically the parts of the wall where slip occurs. Again, both the regularization and augmented Lagrangian approaches can be used [1,13]. In the present work, we use the following regularization of Eq. (1):

$$\tau_w = \tau_c [1 - \exp(-m_c u_w)] + \beta u_w \tag{7}$$

where m_c is a growth parameter similar to the stress growth exponent m of Eq. (6).

Eq. (7) has been tested by Damianou et al. [14] in solving the cessation of Poiseuille flow of a Herschel-Bulkley fluid in a round tube, which is one dimensional. It has also been used to solve the two-dimensional steady-state Poiseuille flow in a rectangular channel [1], giving very satisfactory results for both Newtonian and Bingham flows in those intermediate regimes where wall slip is partial, i.e. it occurs only along a part of the wall around the symmetry plane. The numerical results also agreed with the analytical solution of the Newtonian flow, in regimes where such a solution is available (no wall slip or slip everywhere along the walls).

An interesting observation in the case of the cessation of viscoplastic Poiseuille flow in a tube with wall slip with zero slip yield stress (i.e. $\tau_c = 0$) is that the velocity becomes and remains uniform before complete cessation [1]. Moreover, the theoretical stopping time may become infinite whereas in the absence of slip this is finite. Damianou et al. [14] employed a power-law generalization of the Navier condition

$$\tau_w = \beta u_w^s \tag{8}$$

and showed that the stopping time is finite only when the exponent s is less than unity; otherwise, the stopping time is infinite for any non-zero Bingham number and the volumetric flow rate decays exponentially. However, if the slip yield stress is non-zero, slip ceases at a finite critical time and cessation is accelerated so that the stopping times are finite, in agreement with theoretical estimates [15,16].

The literature concerning solutions of the steady-state viscoplastic flow in rectangular ducts has been reviewed in [1]. In particular, Roquet and Saramito [13] identified the various steady-state regimes observed when the yield stress and the slip yield stress vary and the slip coefficient is fixed. Time-dependent Bingham flows in ducts of various cross-sections with no wall slip have been studied by Muravleva and Muravleva [17] who considered both start-up and cessation flows. The calculated stopping times for the latter flows were found to be in good agreement with the theoretical estimates.

The objective of the present work is to investigate the effect of wall slip on the cessation flow of a Bingham plastic in a square duct. To our knowledge, this flow has not been investigated before. Moreover, it provides a good test for the regularizations of both the Bingham constitutive equation and the slip equation we employ. The rest of the paper is organized as follows. In Section 2, the governing equations of the flow are presented. In Section 3, we provide analytical solutions for the Newtonian flow in the cases of no wall slip, Navier slip, and slip with nonzero slip yield stress. In the latter case, the flow is amenable to analytical solution only below a first and above a second critical value of the pressure gradient. Below the first critical value, the classical no-slip time-dependent solution applies. Above the second one, non-uniform slip occurs everywhere along the wall and the corresponding analytical solution is valid only until slip at the duct corner ceases and thus slip along the wall is partial (nonlinear) thereafter. In Section 4, we present numerical solutions of the Newtonian flow in all the flow regimes. The numerical results coincide with the analytical ones in all regimes where the latter solutions are available. In Section 5, results for the Bingham flow are presented and the no-slip, Navier-slip, and non-zero-slip-yield-stress cases are discussed. It is shown that when Navier slip applies, i.e. when the slip yield stress is zero, the fluid slips at all times and the velocity becomes and remains flat till complete cessation. The evolution of the flat velocity is solved analytically. Interestingly, numerical difficulties are observed in this flow regime, since the regularization becomes problematic when the rate of deformation is zero almost everywhere (but not very close to the wall). Finally, in Section 6 the main conclusions of this work are summarized.

2. Governing equations

We consider the transient Poiseuille flow of a Bingham plastic in a duct of square cross-section and infinite length with $-H \le y \le H$, $-H \le z \le H$, where H is the half-width of the duct. Due to symmetry, only the first quadrant is considered. The flow is governed by the momentum equation, which, under the assumption of negligible gravity, is simplified to

$$\rho \frac{\partial u_x}{\partial t} = G + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{9}$$