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Size-dependent fluid dynamics with application to lid-driven cavity flow

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ABSTRACT

Some physical experiments exhibit size-dependency for fluid flows at small scales. This in turn necessitates the introduction of couple-stresses in the corresponding continuum theory. The resulting size-dependent couple stress fluid mechanics can be used to explore a range of such non-Newtonian flow behavior at micro- and nano-scales, and also to bridge between atomistic and classical continuum theories. Here we concentrate on two-dimensional flow and examine the effects of couple-stresses by developing and then applying a stream function-vorticity computational fluid dynamics formulation. Details are provided both on the governing equations for size-dependent flow and on the corresponding numerical implementation. Afterwards, the formulation is applied to the lid-driven cavity problem to examine the behavior of the flow as a function of the length scale parameter l. The investigation covers a range of Reynolds numbers, and includes an evaluation of the critical value beyond which a stationary response is no longer possible. The additional boundary conditions associated with consistent couple stress theory are found to play an important role in determining the flow pattern and critical Reynolds number.

1. Introduction

Size-dependent responses of fluid flows have been recognized in a range of important physical problems, including those involving biofluids, liquid crystals, lubrication and MEMS devices. On the other hand, the classical Navier–Stokes theory for Newtonian fluids lacks a characteristic length scale, which suggests that a generalization is needed to address adequately such problems exhibiting size-dependency. However, for a theory to be accepted as a generalization of the Navier–Stokes formulation, three stages of verification and validation must be satisfied. Firstly, the theory must be self-consistent, with no indeterminacies. Unless a theory is fully consistent, there is no point to moving forward. Secondly, the theory should be tested on several simple well-known examples, either analytically or computationally, to examine the consequences of the generalization. Finally, the theory must be compared critically with a number of detailed physical experiments designed to test predicted new features of fluid flows. In the present paper, we focus on the second of these stages by developing a computational formulation for consistent size-dependent couple stress fluid dynamics and then by studying in detail lid-driven cavity flows in two-dimensions.

The idea of couple-stresses was first introduced into a theory by Cosserat and Cosserat [1] in continuum mechanics for solids. Several decades later, Toupin [2], Mindlin and Tiersten [3], Koiter [4], Mindlin [5], and Nowacki [6] expanded and generalized the concept. Stokes [7,8] was the first to bring the idea into fluid mechanics. However, the indeterminacy of the spherical part of the couple-stress tensor and the appearance of the body couple in the force-stress relations, both encountered by Mindlin and Tiersten [3], made the formulation inconsistent. All of these difficulties have been resolved recently by discovering the skew-symmetric character of the couple-stress tensor [9,10]. As a result, this fully self-consistent size-dependent couple stress theory includes a characteristic material length l for the fluid that becomes increasingly important as the characteristic geometric dimension of the problem becomes comparable to that level. Interestingly, in the energy equation corresponding to this consistent couple stress theory, there is a couple-stress related term that generates new mechanisms for energy dissipation in the flow. Further details for this new consistent couple stress theory, along with several applications can be found elsewhere [11].

The couple-stress effect makes the system of equations much more complicated than the classical Navier–Stokes equations. As a result, obtaining a general analytical solution is hardly possible. On the other hand, for the case of couple stress creeping flow, an integral representation for two-dimensional boundary value problems and the corresponding boundary element formulation have been developed to obtain numerical solutions for size-dependent

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creeping flow [12]. However, development of boundary element methods for the extension to non-linear flow problems is quite challenging, even for a classical Navier–Stokes formulation [13]. Finite element methods (FEM) are quite suitable for non-linear problems and also can easily model complicated geometrical domains. The initial FEM formulation for the current couple stress theory is mixed because it incorporates variational constraints to reduce continuity requirements. The formulation artificially considers the rotation field to be separate from the velocity field in the underlying functional and then enforces rotation–velocity compatibility via Lagrange multipliers. The corresponding FEM has been developed recently for linear solid mechanics [14,15]. For general non-linear couple stress fluid problems, other more convenient computational fluid dynamics (CFD) methods could be applied. In order to explore the consequences of the couple stress theory in fluids, the finite difference method is perhaps a more appropriate approach, especially for initial studies in simple geometries. Finite difference methods can handle non-linearity and high order derivatives in a straightforward manner, and are particularly well suited for rectangular domains, such as a square cavity.

The lid-driven cavity problem is one of the most common benchmark problems used to verify numerical methods for two-dimensional incompressible flows. From the early works of Kawaguti [16] and Burggraf [17], many researchers [18–22] have studied the flow behavior and the stability of the steady state flow for quite a wide range of Reynolds numbers. In general, these investigations show that when the Reynolds number is high enough, the flow becomes non-stationary, i.e., a stationary solution no longer exists. The critical Reynolds number at which the instabilities start to appear is reported to be within the range of 7500–10,000 [19–22]. However, there are some studies that computed the steady solution at even higher Reynolds numbers [23–26]. Grid refinement, artificial numerical dissipation, and order of accuracy of the numerical scheme are some causes for the discrepancies in the critical value for the classical theory. Of course, there is also the issue of whether a physical experiment for such Reynolds numbers would maintain a two-dimensional flow pattern [27]. We will not address that issue here and instead focus on computational experiments confined to a two-dimensional idealization.

In recent work, Chen et al. [28] and Asadi et al. [29] have examined the driven cavity problem within the framework of micropolar theory, which includes the effects of couple-stresses. In both papers, attention is focused on low Reynolds number flows with $Re = 10$ or less. Clearly, at such levels of Reynolds number in the cavity, flow stability is not an issue. On the other hand, here we employ consistent couple stress theory to investigate the stability of the flow in the two-dimensional lid-driven cavity and examine behavior for different levels of the couple-stress (or length scale) parameter. Furthermore, in this size-dependent non-Newtonian fluid mechanics, additional non-classical boundary conditions are needed relating to rotational degrees of freedom. We consider the two most basic cases, having either zero couple-traction or zero vorticity specified on the boundary. As we shall see, the choice has a dramatic influence on flow patterns and stability.

In the following section, we provide an overview of consistent size-dependent couple stress fluid mechanics, and then in Section 3, the governing equations for plane flow are specialized. Next, we derive the formulation for the lid-driven cavity problem in Section 4. This includes the introduction of the stream function and the non-dimensional form of the governing equations. In Section 5, we develop the numerical methodology and provide details of the formulation for the cavity problem. Numerical results are obtained in Section 6 to verify the formulation and to investigate the effect of couple-stresses on the critical Reynolds numbers at the threshold from stability to initially unstable modes of the flow for both main types of the non-classical boundary conditions noted above. The results are presented for different magnitudes of the length scale parameter and Reynolds numbers. Finally, Section 7 contains a summary and some general conclusions.

2. Couple stress theory for incompressible fluids

In classical fluid mechanics, a force vector represents the total interaction among elements of the fluid. However, in size-dependent fluid mechanics, an additional couple vector is introduced, which accounts for the microstructure of the fluid. This has a significant impact on character of the governing fluid dynamics equations. As a result, within the framework of consistent continuum fluid dynamics, couple-stresses appear as an inevitable consequence of internal interaction of constituents of the fluid, which is associated with the discrete character of matter at the finest scales. Thus, based on this theory, the force–stress tensor is no longer assumed symmetric and the classical Navier–Stokes equations cannot accurately predict the flow characteristics. Thus, size-dependent theory may prove essential to understand the behavior of fluids at micro-scales and to bridge between atomistic and classical continuum theories.

The differential forms of the governing equations of motion for the incompressible fluid in size-dependent couple stress theory are given as [7,10].

Continuity equation:

$$v_{i,i} = 0 \tag{1}$$

Linear equation of motion:

$$\mathrm{T}_{ji,j} + \rho b_i = \rho \frac{Dv_i}{Dt} \tag{2}$$

Angular equation of motion:

$$\mathrm{M}_{ji,j} + \varepsilon_{ijk}\mathrm{T}_{jk} = 0 \tag{3}$$

Here ρ is the constant mass density, v_i is the velocity and b_i is the body force per unit mass in Cartesian coordinates, while ε_{ijk} represents the permutation or Levi–Civita symbol. The tensors T_{ij} and M_{ij} are the true force–stress tensor and pseudo couple-stress tensor, respectively. Note that in this consistent couple stress theory, only body forces are considered and body couples can be decomposed into equivalent body forces and surface-tractions [9,10].

The force-traction vector $t_i^{(n)}$ and couple-traction vector $m_i^{(n)}$ act through a surface element dS with outward directed unit normal vector n_i and are given by

$$t_i^{(n)} = \mathrm{T}_{ji}n_j \tag{4}$$

$$m_i^{(n)} = \mathrm{M}_{ji}n_j \tag{5}$$

Note that we use the same symbol $m_i^{(n)}$ to represent the couple-traction (or moment-traction vector) and its resultant moment.

In size-dependent theory, couple-stresses make the force–stress tensor non-symmetric. More specifically, in consistent size-dependent theory, the pseudo couple-stress tensor M_{ij} is skew-symmetric, i.e.

$$\mathrm{M}_{ji} = -\mathrm{M}_{ij} \tag{6}$$

This makes the couple-traction vector $m_i^{(n)}$ tangent to the surface, which then has only bending effects on the element surface. Due to the skew-symmetry of the couple-stress tensor, a polar couple-stress vector m_i dual to the tensor M_{ij} can be defined as

$$\mathrm{m}_i = \frac{1}{2}\varepsilon_{ijk}\mathrm{M}_{kj} \tag{7}$$