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Short Communication

Start-up flow of a Bingham fluid between two coaxial cylinders under a constant wall shear stress

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ABSTRACT

The analytical solution of the start-up flow of a Bingham fluid between two co-axial cylinders is presented in this study. We focus on that all fluid is at rest initially, then a constant shear stress is exerted by the inner cylinder while the outer remains stationary. The problem is solved using methods of Laplace transform and numerical integration. The unsteady solutions of shear stress and angular velocity as well as the motion of yielding surface are given. We also conduct the limiting analysis at infinitesimal values of the time after initiation.

1. Introduction

A Bingham-plastic [1] is a viscoplastic fluid characterized by yield stress. Different from a Newtonian fluid, a Bingham plastic flows like a fluid if the shear stress in the fluid is greater than yield stress. But, it behaves like a solid if the shear stress is lower than yield stress. For the simple Bingham constitutive relation, the discontinuity at zero strain-rate usually introduces difficulty for analytical or even numerical analysis. Therefore, the analytical analysis of unsteady or even steady flow of a Bingham fluid in a complex geometry has still been a challenging topic by now.

The circular Couette flow of a Bingham fluid has been quite studied over decades, e.g., the steady flow under different forcings [2–4], unsteady motion under different initial conditions or forcings [5,6], the famous problem of Taylor–Couette stability [7–9], and the start-up flow in a pipe or coaxial cylinders [10–12]. Recently, the start-up flow of a thixotropic Herschel–Bulkley fluid in a Couette rheometer has been investigated using the inverse finite element method [13], which is a practical numerical method to calculate the time-dependent motion of yielding surface. Due to the non-linear constitutive equation, most of the unsteady problems were solved with the help of numerical method except [12], in which the Laplace transform method was applied to solve the start-up flow problem. However, the solution may be ill-posed as the time-dependent yielding surface was used as a boundary condition (abbreviated as BC) for solving the free moving boundary problem. But this methodology appears to be a potential tool for analysis. Besides, all the researches above only focus on the forcing of a rotation velocity at one boundary or a given pressure gradient in axial direction, the wall shear stress has not been considered yet. Therefore, this study intends to propose an analytical solution of the circular Couette start-up flow of a Bingham fluid under a constant wall shear stress, which has not been discussed yet so far.

In our problem, a Bingham fluid filled between two co-axial cylinders is at rest initially. After initiation, the inner cylinder rotates to exert a constant shear stress, and the outer remains stationary. The cylinders are assumed to be placed vertically and infinitely long, so that end effect is neglected. In particular, we reasonably assume the shear stress at the outer wall is finite and less than yield stress. Taking advantage of the assumed BC, we can apply Laplace transform method to solve our problem without using the free moving BC of yielding surface. The unsteady shear stress and velocity as well as the motion of yielding surface are solved with the help of numerical methods of integration and root finding. The unsteady solutions are validated with steady-state one. We also discuss the evolution of shear layer thickness at infinitesimal values of the time.

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2. Formulation

2.1. Governing equations and boundary conditions

As shown in Fig. 1, the two-dimensional cylindrical coordinate system is used with r-axis and θ-axis along the radial and angular directions. The counterclockwise direction of θ is defined to be positive. Since perpendicular to (r,θ)-plane the gravitational effect is not considered hereafter. The flow motion is assumed to be symmetric, so that all variables are independent of θ. Due to the stationary outer cylinder, the radial velocity is zero in the gap. Therefore, the simplified equation of momentum conservation in θ-direction reads

$$\rho\frac{\partial u_\theta}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{\theta r}), \tag{1}$$

where ρ is constant fluid density; u_θ is angular velocity; $\tau_{\theta r}$ is shear stress in θ-direction along the radial axis. In our problem, the only forcing is a constant shear stress in the counterclockwise direction on the inner cylinder wall. The strain rate along radial direction must be positive. Due to the viscous effect, as only the inner cylinder rotates, the flow has a maximum speed at the inner wall, and remains stationary near the outer. Therefore, the direction of angular velocity must be in clockwise; namely, $u_\theta < 0$. Thus, the constitutive relation between shear stress and shear rate tensor [2] is further simplified as

$$\tau_{\theta r}=\tau_0+\mu r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right),\ \text{for } \tau_{\theta r}\geqslant\tau_0, \tag{2}$$

$$\left|r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right|=0,\ \text{for } \tau_{\theta r}<\tau_0, \tag{3}$$

where τ_0 and μ are constant yield stress (unit: $\mathrm{N/m^2}$) and dynamic viscosity (Pa s), respectively. Eq. (2) represents the constitutive relation for shear layer and Eq. (3) is for plug layer.

Due to the non-linearity of Bingham model, at the interface between shear and plug layers, or called yielding surface, velocity must be continuous but the slope of velocity profile may not be. As the shear layer thickness is defined as δ, the yielding surface is located at $r = R_1 + \delta$. Since this location of interface is also an unknown, we need one extra condition at it. So that we impose the BC that shear stress equals yield stress

$$\left|r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right|=0 \text{ and } |\tau_{\theta r}|=\tau_0,\ \text{at } r=R_1+\delta, \tag{4a, b}$$

and velocity is continuous (but not the slope of velocity profile)

$$u_\theta|_{\text{plug layer}}=u_\theta|_{\text{shear layer}},\ \text{at } r=R_1+\delta, \tag{5}$$

where $\delta\in[0,R_2-R_1]$. As the direction of shear stress is counterclockwise, Eq. (4b) can be simplified as $\tau_{\theta r}=\tau_0$ at $r=R_1+\delta$. The BC of a positive constant shear stress τ_w at the inner wall is imposed

$$\tau_{\theta r}=\tau_w,\ \text{at } r=R_1, \tag{6}$$

and the no-slip BC at the outer wall as

$$u_\theta=0,\ \text{at } r=R_2. \tag{7}$$

Besides, since the forcing is from the inner wall, the maximum values of shear stress shall exists on it. Hence, the shear stress at the outer wall must be less than the forcing. Therefore, at the outer wall an additional condition is imposed

$$\tau_{\theta r}<\tau_w,\ \text{at } r=R_2. \tag{8}$$

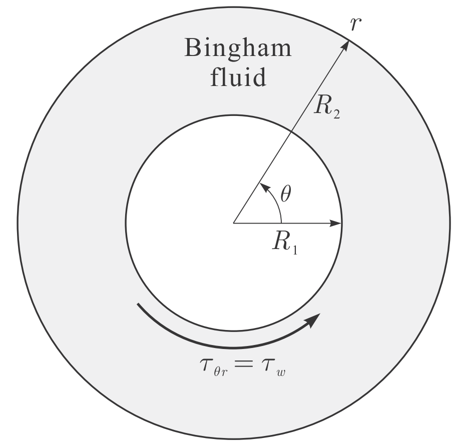

Fig. 1. Definition sketch of two cylinders and the coordinate system. The clockwise direction of θ is defined as positive. The radii of the inner and outer cylinders are R_1 and R_2, respectively. A constant shear stress τ_w at the inner wall is exerted counterclockwise after initiation, and the outer remains stationary.

2.2. Normalization

Non-dimensional variables are defined as follows

$$\tau_{\theta r}=\tau_w\tau^*_{\theta r},\quad u_\theta=(\tau_wR_1/\mu)u^*_\theta,\quad t=\left(\rho R_1^2/\mu\right)t^*,\ \text{and } r=R_1r^*, \tag{9}$$

where variables with asterisk represent the normalized one. Since shear layer thickness δ changes with time, it is also considered as a variable. Normalized shear layer thickness is defined as

$$\beta=\delta/R_1\in[0,R_2/R_1-1]. \tag{10}$$

With Eq. (10) the normalized location of the yielding surface is

$$r^*=\Delta=1+\beta. \tag{11}$$

Two important normalized parameters are given by

$$\alpha=R_2/R_1 \text{ and } B=\tau_w/\tau_0. \tag{12}$$

The parameter α, called *radius ratio*, denotes the normalized location of the outer cylinder. It also represents the ratio of radii of outer to inner cylinders and $\alpha>1$. The parameter B, normalized inner-cylinder wall shear stress, represents the ratio of shear stress on the inner wall to yield stress. B must be greater than yield stress; namely, $B=\tau_w/\tau_0>1$, otherwise the fluid will remain stationary.

We use all normalized variables hereafter, and omit all asterisks on them for clarification. With Eq. (9) the normalized momentum equation becomes

$$\frac{\partial u_\theta}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{\theta r}), \tag{13}$$

and the normalized constitutive relation are

$$\tau_{\theta r}=\frac{1}{B}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right),\ \text{for } \tau_{\theta r}\geqslant 1/B, \tag{14}$$

$$\left|r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right|=0,\ \text{for } \tau_{\theta r}<1/B. \tag{15}$$

Finally, the normalized BCs are

$$\tau_{\theta r}=1,\ \text{at } r=1, \tag{16}$$

at the inner wall, and

$$u_\theta=0 \text{ and } \tau_{\theta r}<1,\ \text{at } r=\alpha. \tag{17}$$

at the outer wall, and