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An exact analytical solution for viscoelastic fluids with pressure-dependent viscosity

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ABSTRACT

A linear relationship between the shear viscosity and the total pressure, a constant single relaxation time for a Maxwell-type viscoelastic fluid, and a unidirectional velocity profile are the major assumptions made in the present work in order to study the steady-state isothermal and pressure-driven flows in straight channels and circular tubes. Despite their non-linearity the final partial differential equations that govern the flows are solved analytically, and the dependence of all the primary flow variables on the geometrical aspect ratio, the dimensionless pressure-viscosity coefficient and the Weissenberg number is revealed explicitly. It is demonstrated that the pressure-dependent viscosity slightly affects the velocity profile, changes substantially the pressure gradient along the main flow direction, generates another normal to the wall, and it is responsible for significant variations of the extra-stresses along both spatial directions. An exponential increase of the viscosity, relative to its reference value, is predicted as the distance from the exit of the channel/tube increases. As a consequence, the average pressure difference, required to drive the flow and the shear stress at the wall increase substantially compared to that predicted by the classic Hagen–Poiseuille law. Last, it is revealed that the solution of the governing equations ceases to exist when the Weissenberg number reaches a threshold.

1. Introduction

As a first approximation, the shear viscosity, η^*, for a big variety of liquids is assumed constant. In general however, it depends on the flow conditions and in particular on the shear-rate, the pressure, and, for non-isothermal problems, strongly on the temperature. The effect of the pressure on the viscosity becomes important at a pressure 50 atm approximately [1], while for pressures of the order of 1000 atm the viscosity appears to increase more than an order of magnitude [2,3]. Applications which involve a high pressure difference and/or a large pressure range include polymer processing operations such as extrusion and injection molding [1,4–6], food processing, pharmaceutical tablet manufacturing, crude oil and fuel oil pumping [7], fluid film lubrication [8], journal bearing applications [9], microfluidics [10] and geophysics [11]. In order to avoid major errors when modeling these types of processes the dependence of the viscosity on the pressure must be taken into account.

Stokes [12] was the first to introduce the concept of the pressure-dependent shear viscosity, and indeed early experiments by Barus [13,14], later by Bridgman [15] and Griest et al. [16], and more recently by Iqbal and Hasmi [17], Bair et al. [18], Bair and Kottke [19], Pruša et al. [20] and others have clearly demonstrated the increase of the viscosity with the increase in pressure p^*. Experimental methods and techniques for measuring the shear viscosity versus pressure can be found in many papers in the literature such as those by Kadijk and Van Den Brule [5], Binding et al. [21], Goubert et al. [22], Park et al. [23] and Schaschke [24].

As far as the dependence of the viscosity on the pressure is concerned, it appears that a relationship between η^* and p^* which can describe adequately all the available experimental data does not exist. Indeed, at low to medium pressure differences a linear law, first proposed by Barus [13,14], is suitable and often used by many researches:

$$\eta^* = \eta_0^* \left[1 + \beta^* (p^* - p_{ref}^*)\right] \tag{1}$$

where β^* is the constant pressure-viscosity coefficient, and η_0^* is the shear viscosity at the reference pressure p_{ref}^*; throughout the text a superscript * denotes a dimensional quantity. At large pressure differences an exponential law appears to fit the data pretty well, while for huge pressure differences the increase is even larger than exponential [18,19]. For details on the empirical relationships $\eta^* = \eta^*(p^*)$ the interesting reader is referred to the works of Bair

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and Kottke [19], Huilgol and You [25], Málek and Rajagopal [26] and Hron et al. [27].

Regarding the modeling of fluids with pressure-dependent viscosity, a major problem should be reported. In particular, with some laws which appear to describe the experimental data well (for a specific range of the pressure) such as the exponential law, it is very difficult to solve the governing equations analytically or even numerically [28]. An approximate analytical solution for the adiabatic capillary flow of a fluid with pressure- and temperature-dependent viscosity using the exponential law, has been derived by Denn [29], however in many cases the corresponding governing equations do not have a solution even for simple flows, such as steady-state pressure driven and Couette type flows, or a solution cannot be found [25,27].

Typical values for β^* are 10–50 GPa^{-1} for polymer melts [1,5,6,30–32], 10–70 GPa^{-1} for lubricants [18,33] and 10–20 GPa^{-1} for mineral oils [34]. These values for β^* are usually reported for the exponential law; however, they are also valid for small or medium pressure differences $p^* - p^*_{ref}$.

The pressure-dependence of the viscosity in lubrication [8], viscometric and other flows has been analyzed mathematically by various investigators [2,27,35–37]. Numerical simulations for generalized shear-thinning Newtonian liquids have been performed by Lanzendörfer [28], Lanzendörfer and Stebel [38] and Hirn et al. [39]. Asymptotic solutions of weakly compressible Newtonian Poiseuille flows with pressure dependent viscosity have been derived by Poyiadji et al. [40], while the effect of the pressure-dependent viscosity for the unbounded flow past a sphere has been studied by Housiadas et al. [41].

However, studies for viscoelastic liquids with pressure-dependent viscosity are absent from the literature with the exception of the works by Karra et al. [42] and Housiadas [43]. In particular, Karra et al. [42] studied the transient unidirectional flow between a stationary and an oscillatory plate using the Maxwell constitutive model [44] and assuming that the shear viscosity and the relaxation time vary both linearly, or both exponentially, with pressure. Their results showed a few interesting phenomena associated with the pressure-dependent viscosity and relaxation time of the fluid. Housiadas [43] used the exponential law along with an exponential dependence of the relaxation time on the pressure and utilized asymptotic techniques in order to find the solution of the relevant non-linear governing equations (see more comments below).

In the present work, Eq. (1) is used in conjunction with the Maxwell constitutive model for the description of the viscoelastic response of the fluid under deformation (typically a polymer melt), in order to study the steady-state isothermal and pressure-driven flows in channels/slits and circular tubes. Eq. (1) is valid for all fluids for sufficient small values of $\beta^*(p^* - p^*_{ref})$ and therefore it can be used to reveal the lower limit effect of the pressure-dependent viscosity on the flow field. As far as the single relaxation time of the fluid, λ^*, is concerned, it is known that λ^* is given as the ratio of the shear viscosity to the shear modulus. However, the dependence of λ^* on p^* is not clear and, surprisingly, experimental data for λ^* versus p^* do not exist in the literature (at least as far as the author this work is aware). Handge and Altstadt [32] have suggested that if the time-temperature–pressure superposition principle is fulfilled, which appears to be the case for the polystyrene melts that they report on, then all relaxation times must scale with the same factor as the viscosity. In the present paper however λ^* is assumed to be constant; as shown in Ref. [43] this assumption does not qualitatively affect the results. Although the linear pressure-dependent viscosity and constant relaxation time are major assumptions, they allow solving analytically the non-linear two-dimensional governing equations for the pressure-driven flows mentioned above, reveal interesting observations, and highlight a few problems with the modeling of flows with the pressure-dependent viscosity such as the loss of existence of solution at high Weissenberg numbers.

Finally, the differences between this work and that by Housiadas [43] should be explicitly stated. Here, the shear viscosity of the fluid depends linearly on the pressure, the relaxation time is constant and unidirectional flow is assumed, i.e. wall-normal motion of the fluid is not allowed. The solutions of the final non-linear partial differential equations that govern the flow problems are exact; hence they hold for arbitrary values of the dimensionless number and parameters which appear in the equations. In Ref. [43], the shear viscosity and the relaxation time depend exponentially on the pressure, the velocity profile depends on both the axial and the wall-normal directions, and the solutions have been found up to sixth order using perturbation methods with small parameter the dimensionless pressure-viscosity coefficient; this implies that the solutions reported in Ref. [43] are valid for small values of the dimensionless numbers and parameters. Even if special care on the accuracy and convergence of the asymptotic solutions was given, uncertainties which concern with the range of the parameters for which the solution is valid always exist with asymptotic methods.

The rest of the present paper is composed of four sections. In Section 2, the problem formulation, assumptions, governing equations and accompanying boundary conditions are stated. In Section 3, the analytical solutions for both geometrical flow configurations (in straight channels/slits and in circular tubes) are provided. Results and discussion are given in Section 4, and concluding remarks, in Section 5, finalize the paper.

2. Governing equations

The isothermal, steady, and pressure-driven flows of an incompressible viscoelastic fluid with constant mass density ρ^*, in two different geometries, a straight channel (or slit) of length L^* and height $2R^*$, and a circular tube of length L^* and constant radius R^*, are considered (see Fig. 1). In absence of external forces, neglecting gravity and taking into account that for slow flows of highly viscous polymer melts inertia is negligible, the mass and momentum equations are:

$$\nabla^* \cdot \mathbf{v}^* = 0 \tag{2}$$

$$-\nabla^* p^* + \nabla^* \cdot \boldsymbol{\tau}^* = \mathbf{0} \tag{3}$$

In Eqs. (2) and (3), $\mathbf{v}^* = \mathbf{e_z} v_z^* + \mathbf{e_r} v_r^*$ is the velocity vector, with $\mathbf{e_z}$, $\mathbf{e_r}$ being the unit vectors, p^* is the total pressure, and $\boldsymbol{\tau}^*$ is the polymer

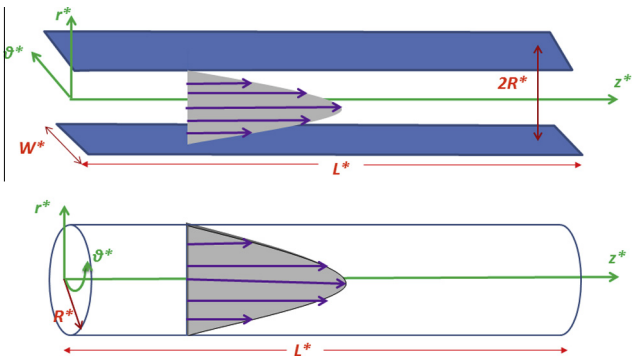

Fig. 1. Geometrical flow configurations and coordinate systems; top: flow between two parallel walls, bottom: flow in a straight tube.