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Linear instability of a highly shear-thinning fluid in channel flow

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ABSTRACT

We study pressure-driven channel flow of a simple viscoelastic fluid whose elastic modulus and relaxation time are both power-law functions of shear-rate. We find that a known linear instability for the case of constant elastic modulus (Wilson and Rallison, 1999) persists and indeed becomes more dangerous when the elastic modulus is allowed to vary. The most unstable scenario is a highly shear-thinning relaxation time with a slightly shear-thinning elastic modulus, and typical unstable perturbations have a wavelength comparable with the channel width. Inertia is mildly destabilising.

We compare with microchannel experiments (Bodiguel et al., 2015), and find qualitative agreement on the critical flow rate for instability; however, because of the artificial nature of the power-law viscosity, we have excluded the sinuous modes of instability which are seen in experiment.

1. Introduction

It is well known [3] that viscoelastic fluids can exhibit instabilities not seen in their Newtonian counterparts. Where such an instability persists in the absence of inertia, it is termed an *elastic instability*.

Perhaps the most well-understood elastic instability is the *curved streamline instability* discovered by Larson et al. [4] and elucidated by Pakdel and McKinley [5]. Here the first normal stress difference interacts with curvature of the streamlines to drive an instability.

Another broad category of elastic instabilities is *interfacial instabilities*. A jump in material properties across an interface can trigger instability of the interface: either through a long-wave mechanism based on the tilting of the interface [6] or in some cases [7] by a mechanism that remains obscure (and persists even when surface tension holds the interface flat) but nonetheless depends critically on the presence of the interface.

A third category is *shear-banding instabilities*: a fluid whose constitutive curve is non-monotonic may spontaneously form bands of different shear stress (in a rate-controlled scenario) or different shear rate (in a stress-controlled scenario) [8]. This seemingly unphysical behaviour does seem to occur for real physical systems [9] and has been the focus of much recent work [10].

However, recently published experiments by Bodiguel et al. [2] have found evidence of an elastic instability, occurring at a reproducible critical flow rate, in a flow having neither curved streamlines, nor an interface, nor any evidence of shear-banding. There is, to our knowledge, only one theoretical prediction of such an instability, in a study by Wilson and Rallison [1]. In this paper we extend that analysis to a constitutive model which can match the rheometry of the fluid used in experiments. We find an instability whose critical flowrate is reasonably close to that seen in the experiments; but there are limitations to our model.

In Section 2 we introduce our constitutive model and show its behaviour in simple shear flow; in Section 3 we carry out a linear stability analysis for channel flow of this new fluid. In Section 4 we present the results of our study, including the dependence of the instability on fluid parameters, on inertia and on perturbation wavenumber. In Section 5 we make a detailed comparison with the experimental results published in [2]; finally in Section 6 we draw our overall conclusions.

2. Model fluid

Our model fluid is chosen with three principles in mind. It needs to match the rheometry of the experiments we wish to replicate [2]; some limit of it needs to match the existing theory [1]; and it is desirable for it to have at least a semi-physical microscopic derivation.

In the experiments, the fluid is highly shear-thinning. The rheometry shows that both the viscosity and first normal stress difference are reasonably well fit with a simple power law over a good range of shear rates. Thus we need a viscoelastic model whose parameters can vary with shear rate.

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Our previous theory [1] used a special case of the White–Metzner model whose relaxation time had power-law dependence on shear rate but whose modulus was independent of flow. We need to extend this fluid to allow a wider range of rheology in the fluid.

All White–Metzner style models are simply phenomenological extensions of the UCM model; UCM, on the other hand, does have a physical derivation as the polymer stress contribution of a dilute solution of Hookean dumbbells (see, for example [11]). The model we will use in this paper comes from a semi-physical extension of the UCM derivation, which is to allow the spring constant and the solvent viscosity to vary with the background shear rate (but without a kinetic theory to explain the behaviour of these two parameters). The derivation produces the following constitutive equation for the extra-stress tensor $\underline{\underline{\tau}}$:

$$\underline{\underline{\tau}} = G(\dot\gamma)\underline{\underline{A}} \qquad \overset{\nabla}{\underline{\underline{A}}} = -\frac{1}{\lambda(\dot\gamma)}(\underline{\underline{A}} - \underline{\underline{I}}) \tag{1}$$

in which the rheological functions G (shear modulus) and λ (relaxation time) depend on the instantaneous shear rate $\dot\gamma$, and $\overset{\nabla}{\underline{\underline{A}}}$ is the upper-convected derivative, defined below in Eq. (6).

This is not equivalent to the standard White–Metzner model, which is given by the following equation:

$$\underline{\underline{\tau}} + \lambda(\dot\gamma)\overset{\nabla}{\underline{\underline{\tau}}} = \eta(\dot\gamma)(\underline{\nabla u} + \underline{\nabla u}^\top)$$

in which $\eta(\dot\gamma)$ is the shear-rate dependent viscosity and $\underline{u}$ the fluid velocity; however, in the special case $\eta(\dot\gamma) = G\lambda(\dot\gamma)$ for constant G, the two models both reduce to the form considered in [1].

2.1. Governing equations

The full governing equations for our incompressible fluid (in the absence of external body forces such as gravity) are:

$$\underline{\nabla} \cdot \underline{u} = 0 \tag{2}$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla u}\right) = \underline{\nabla} \cdot \underline{\underline{\sigma}} \tag{3}$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + G\underline{\underline{A}} \tag{4}$$

$$\overset{\nabla}{\underline{\underline{A}}} = -\frac{1}{\lambda}(\underline{\underline{A}} - \underline{\underline{I}}) \tag{5}$$

$$\overset{\nabla}{\underline{\underline{A}}} \equiv \frac{\partial}{\partial t}\underline{\underline{A}} + \underline{u} \cdot \underline{\nabla \underline{\underline{A}}} - (\underline{\nabla u})^\top \cdot \underline{\underline{A}} - \underline{\underline{A}} \cdot \underline{\nabla u}. \tag{6}$$

Here $\underline{u}$ is the fluid velocity, ρ its density, $\underline{\underline{\sigma}}$ the total stress tensor, p pressure, $\underline{\underline{A}}$ the conformation tensor, G the elastic modulus and λ the relaxation time. $\underline{\underline{I}}$ is the identity tensor (or unit matrix).

The parameters λ and G depend on the shear rate $\dot\gamma$, defined as an invariant of the rate-of-strain tensor $\underline{\underline{E}}$ as follows:

$$\dot\gamma = \sqrt{2\underline{\underline{E}} : \underline{\underline{E}}} \text{ where } \underline{\underline{E}} = \frac{1}{2}(\underline{\nabla u} + \underline{\nabla u}^\top). \tag{7}$$

2.2. Rheometry

We use cartesian coordinates (x, y). In a simple steady shear flow $\underline{u} = \dot\gamma y \underline{e}_x$ the fluid stress is

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p_0 + G + 2G\lambda^2\dot\gamma^2 & G\lambda\dot\gamma \\ G\lambda\dot\gamma & -p_0 + G \end{pmatrix},$$

which gives the two viscometric functions:

$$\eta \equiv \frac{\sigma_{12}}{\dot\gamma} = G\lambda \qquad \Psi_1 \equiv \frac{\sigma_{11} - \sigma_{22}}{\dot\gamma^2} = 2G\lambda^2.$$

As we will see in Section 5, for the fluid used in the experiments it is reasonable, over a range of shear rates, to approximate both η and Ψ_1 with power-law functions of $\dot\gamma$ of the form $A\dot\gamma^n$. This allows us to restrict our model to power-law behaviour for the functions $G(\dot\gamma)$ and $\lambda(\dot\gamma)$:

$$G = G_M\dot\gamma^{m-n} \quad \lambda = K_M\dot\gamma^{n-1} \tag{8}$$

where the indices m and n are chosen so that the definition of λ matches that used in [1], and the shear stress has the simple scaling $\sigma_{12} \sim G_M K_M \dot\gamma^m$.

3. Stability theory

We now consider two-dimensional channel flow of a fluid satisfying Eqs. (2)–(7) along with the scaling laws of Eq. (8). The channel, of infinite extent in the x-direction, has half-height L (in the y-direction) and the flow is driven by a pressure gradient $\mathcal{P}$ in the x-direction.

3.1. Steady channel flow

If we assume a steady, unidirectional flow profile $\underline{u} = U(y)\underline{e}_x$, satisfying a no-slip condition at $y = \pm L$, we obtain the following solution:

$$U(y) = \left(\frac{\mathcal{P}}{G_M K_M}\right)^{1/m}\left(\frac{m}{m+1}\right)(L^{(m+1)/m} - |y|^{(m+1)/m}) \tag{9}$$

$$\dot\gamma = |U'| = \left(\frac{\mathcal{P}|y|}{G_M K_M}\right)^{1/m}. \tag{10}$$

3.2. Dimensionless form

We now convert to dimensionless quantities. Introducing U_0 to denote the centreline velocity, we scale lengths with L, the channel half-width; times with the average shear rate U_0/L; and stresses with a typical shear stress, which is the fluid shear stress σ_{12} at the average shear rate: $G_M K_M (U_0/L)^m$.

Denoting scaled quantities with a tilde, our new governing equations become:

$$\underline{\nabla} \cdot \underline{\tilde u} = 0 \qquad \mathrm{Re}\left(\frac{\partial \underline{\tilde u}}{\partial t} + \underline{\tilde u} \cdot \underline{\nabla \tilde u}\right) = \underline{\nabla} \cdot \underline{\underline{\tilde\sigma}} \tag{11}$$

$$\underline{\underline{\tilde\sigma}} = -\tilde p\underline{\underline{I}} + \frac{C}{\mathcal{W}}\underline{\underline{A}} \qquad \overset{\nabla}{\underline{\underline{A}}} = -\frac{1}{\mathcal{W}}(\underline{\underline{A}} - \underline{\underline{I}}) \tag{12}$$

along with the definitions of $\overset{\nabla}{\underline{\underline{A}}}$ and $\dot\gamma$ which are unchanged (save for the addition of tildes) from their original form in Eqs. (6) and (7). (Note that $\underline{\underline{A}}$ was dimensionless in the original equations and has not been scaled.)

We have introduced the Reynolds number (based on the shear viscosity at the average shear rate)

$$Re = \frac{\rho U_0 L}{G_M K_M (U_0/L)^{m-1}} \tag{13}$$

and the new viscometric functions