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Flow of a generalised Newtonian fluid due to a rotating disk

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ABSTRACT

The boundary-layer flow due to a rotating disk is considered for a number of generalised Newtonian fluid models. In the limit of large Reynolds number the flow inside the three-dimensional boundary-layer is determined via a similarity solution. Results for power-law and Bingham plastic fluids agree with previous investigations. We present solutions for fluids that adhere to the Carreau viscosity model. It is well known that unlike the power-law and Bingham models the Carreau model is applicable for vanishingly small, and infinitely large shear rates, as such we suggest these results provide a more accurate description of non-Newtonian rotating disk flow.

1. Introduction

The steady incompressible flow induced by the rotation of an infinite plane with uniform angular velocity is an exact solution of the Navier–Stokes equations, as was first described by von Kármán [1]. The flow is characterised by the lack of a radial pressure gradient near to the disk to balance the centrifugal forces so the fluid spirals outwards. The disk acts as a centrifugal fan, the fluid emanating from the disk being replaced by an axial flow directed back towards the surface of the disk.

Batchelor [2] showed that this type of flow is in fact just a limiting case of a whole number of flows with similarity solutions in which both the infinite plane and the fluid at infinity rotate with differing angular velocities. The corresponding limiting case, when the infinite plane is stationary and the fluid at infinity rotates at a constant angular velocity, was first described by Bödewadt [3].

A vast wealth of material exists concerning the solutions of the Newtonian rotating disk equations; the interested reader is referred to Zandbergen and Dijkstra [4]. The authors provide a thorough review of the major contributions made postdating von Kármán's seminal work.

Considerably less attention has been given to the corresponding non-Newtonian rotating disk problem. Mitschka [5] modified the von Kármán similarity solution to incorporate a power-law governing viscosity relationship. In this case the base flow is not an exact solution of the generalised Navier–Stokes equations and a boundary-layer approximation is required. Both Mitschka and Ulbrecht [6] and Andersson et al. [7] present basic flow solutions for shear-thickening and shear-thinning power-law fluids. However, the authors overlooked the importance of matching these boundary-layer solutions to an external flow. Denier and Hewitt [8] addressed this problem and presented corrected solutions for both cases, noting that the structure of the solutions is intrinsically different for shear-thickening and shear-thinning fluids.

More recently, Ahmadpour and Sadeghy [9] (subsequently referred to herein as AS) formally addressed the problem of the flow due to a rotating disk when one considers Bingham plastic fluids. Claiming to have found an exact solution to the problem, the authors are only able to present numerical solutions for specific values of the Reynolds number (Re) and dimensionless radius of the disk (r). Having not considered the boundary-layer formulation of the problem, the authors find that terms dependent on Re and r appear in the formulation of the governing base flow ODEs, and thus have the need to provide specific values for these constants during their numerical solution process.

In this study we determine steady mean flow solutions for power-law, Bingham and Carreau fluid models. The power-law results are essentially a review of the work of Denier and Hewitt [8] but are included here as they prove useful to compare with the results owing from the more complex Carreau model. By introducing the modified Bingham number used by Matsumoto et al. [10] in their film thickness investigation, we are able to determine a governing set of ODEs dependent solely on this parameter, these results are then compared to those of AS. Additionally, we present solutions for shear-thickening and shear-thinning Carreau fluids where now the flow is controlled by not one, but three

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dimensionless parameters. In Section 2 we formulate the problem in the general case, results are presented in Section 3 and are discussed in Section 4. Conclusions are drawn in Section 5.

2. Formulation

Consider the flow of a steady incompressible generalised Newtonian fluid due to a rotating disk located at $z^*=0$. The disk rotates about the z^*-axis with angular velocity Ω^*. Working in a reference frame that rotates with the disk, the continuity and Cauchy momentum equations are expressed as

$$\nabla\cdot\boldsymbol{u}^*=0, \tag{1a}$$

$$\rho^*[\boldsymbol{u}^*\cdot\nabla\boldsymbol{u}^*+\boldsymbol{\Omega}^*\times(\boldsymbol{\Omega}^*\times\boldsymbol{r}^*)+2\boldsymbol{\Omega}^*\times\boldsymbol{u}^*]=-\nabla p^*+\nabla\cdot\boldsymbol{\tau}^*. \tag{1b}$$

Here $\boldsymbol{u}^*=(u^*,v^*,w^*)$ are the velocity components in cylindrical polar coordinates (r^*,θ,z^*), the angular velocity vector is $\boldsymbol{\Omega}^*=(0,0,\Omega^*)$, the position vector is $\boldsymbol{r}^*=(r^*,0,z^*)$, the fluid density is ρ^* and p^* is the fluid pressure. The stress tensor $\boldsymbol{\tau}^*$ for incompressible generalised Newtonian fluids is given by

$$\boldsymbol{\tau}^*=\mu^*\dot{\boldsymbol{\gamma}}^*\quad\text{with}\quad\mu^*=\mu^*(\dot{\gamma}^*), \tag{2}$$

where $\dot{\boldsymbol{\gamma}}^*=\nabla\boldsymbol{u}^*+(\nabla\boldsymbol{u}^*)^{\mathrm{T}}$ is the rate-of-strain tensor and $\mu^*(\dot{\gamma}^*)$ is the generalised Newtonian viscosity. The magnitude of the rate-of-strain tensor is $\dot{\gamma}^*=\sqrt{(\dot{\boldsymbol{\gamma}}^*:\dot{\boldsymbol{\gamma}}^*)/2}$. The governing relationships for $\mu^*(\dot{\gamma}^*)$ that will be considered herein are:

$$\text{Power-law model}\quad-\quad\mu^*=m^*(\dot{\gamma}^*)^{n-1}, \tag{3a}$$

$$\text{Bingham model}\quad-\quad\mu^*=\begin{cases}\infty & \text{for } \tau^*<\tau_y^*,\\ \mu_p^*+\tau_y^*(\dot{\gamma}^*)^{-1} & \text{for } \tau^*\geqslant\tau_y^*,\end{cases} \tag{3b}$$

$$\text{Carreau model}\quad-\quad\mu^*=\mu_\infty^*+(\mu_0^*-\mu_\infty^*)[1+(\lambda^*\dot{\gamma}^*)^2]^{(n-1)/2}. \tag{3c}$$

Here m^* is the consistency coefficient and n is the fluid index, for $n>1$ the fluid is said to be shear-thickening, whilst for $n<1$ the fluid is said to be shear-thinning. The Newtonian viscosity relationship is recovered when $n=1$, $\tau_y^*=0$ and $\mu_0^*=\mu_\infty^*$, respectively. The plastic-shear-rate viscosity is μ_p^*, the magnitude of the shear stress tensor is $\tau^*=\sqrt{(\boldsymbol{\tau}^*:\boldsymbol{\tau}^*)/2}$ and τ_y^* is the yield stress. The infinite-shear-rate viscosity is μ_∞^*, the zero-shear-rate viscosity is μ_0^* and λ^* is the characteristic time constant, often referred to as the 'relaxation time'.

Assuming the flow to be axisymmetric the components of the stress tensor are

$$\tau_{r^*r^*}=2\mu^*\left(\frac{\partial u^*}{\partial r^*}\right), \tag{4a}$$

$$\tau_{\theta\theta}=2\mu^*\left(\frac{u^*}{r^*}\right), \tag{4b}$$

$$\tau_{z^*z^*}=2\mu^*\left(\frac{\partial w^*}{\partial z^*}\right), \tag{4c}$$

$$\tau_{r^*z^*}=\mu^*\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)=\tau_{z^*r^*}, \tag{4d}$$

$$\tau_{r^*\theta}=\mu^*\left[r^*\frac{\partial}{\partial r^*}\left(\frac{v^*}{r^*}\right)\right]=\tau_{\theta r^*}, \tag{4e}$$

$$\tau_{\theta z^*}=\mu^*\left(\frac{\partial v^*}{\partial z^*}\right)=\quad\tau_{z^*\theta}, \tag{4f}$$

where the magnitude of the rate-of-strain tensor takes the form

$$\dot{\gamma}^*=\left\{2\left[\left(\frac{\partial u^*}{\partial r^*}\right)^2+\left(\frac{u^*}{r^*}\right)^2+\left(\frac{\partial w^*}{\partial z^*}\right)^2\right]+\left[r^*\frac{\partial}{\partial r^*}\left(\frac{v^*}{r^*}\right)\right]^2+\left(\frac{\partial v^*}{\partial z^*}\right)^2+\left(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}\right)^2\right\}^{1/2}. \tag{5}$$

In the rotating frame of reference this system is closed subject to the boundary conditions

$$u^*=v^*=w^*=0\quad\text{at}\quad z^*=0, \tag{6a}$$

$$u^*\to0,\quad v^*\to-r^*\Omega^*\quad\text{as}\quad z^*\to\infty. \tag{6b}$$

We non-dimensionalise the system by writing

$$u(r,z)=\frac{u^*(r^*,z^*)}{l^*\Omega^*},\quad v(r,z)=\frac{v^*(r^*,z^*)}{l^*\Omega^*},\quad r=\frac{r^*}{l^*},$$

$$w(r,z)=\frac{w^*(r^*,z^*)}{\delta^*l^*\Omega^*},\quad p(r,z)=\frac{p^*(r^*,z^*)}{\rho^*(l^*\Omega^*)^2},\quad z=\frac{z^*}{\delta^*l^*}.$$

We note here that the axial coordinate and velocity component have been scaled by the boundary-layer thickness, δ^*, this is in anticipation of a boundary-layer structure arising on the rotating disk. One finds that

$$\delta^*=Re^{-1/(q+1)}\quad\text{with}\quad Re=\frac{\rho^*{\Omega^*}^{2-q}{l^*}^2}{\sigma^*}, \tag{7}$$

where throughout the forthcoming analysis $q=n$ for power-law fluids and $q=1$ for Bingham plastic and Carreau fluids, whilst $\sigma^*=m^*,\mu_p^*,\mu_\infty^*$ for power-law, Bingham plastic and Carreau fluids, respectively. Thus, the scaled governing equations are

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}=0, \tag{8a}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{(v+r)^2}{r}=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+\frac{1}{Re^{2/(q+1)}}\left\{\frac{2}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial r}\right)-\frac{2\mu u}{r^2}\right\}, \tag{8b}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+2u=\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+\frac{1}{Re^{2/(q+1)}}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[\mu r^3\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]\right\}, \tag{8c}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-Re^{2/(q+1)}\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial z}\right)+2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)+\frac{1}{Re^{2/(q+1)}}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial w}{\partial r}\right)\right\}, \tag{8d}$$

where the dimensionless viscosity functions (in the yielded region when considering Bingham plastic fluids) μ are defined as

$$\text{Power-law model}\quad-\quad\mu=(\hat{\mu})^{n-1}, \tag{8e}$$

$$\text{Bingham model}\quad-\quad\mu=1+2rB_n(\hat{\mu})^{-1}, \tag{8f}$$

$$\text{Carreau model}\quad-\quad\mu=1+c_0[1+(k\hat{\mu}/r)^2]^{(n-1)/2}, \tag{8g}$$

where $B_n=\tau_y^*/\left(2r^*\Omega^*\sqrt{\mu_p^*\rho^*\Omega^*}\right)$ is the Bingham number defined by Matsumoto et al. [10]. The authors experimental investigations have shown this quantity to be $\mathcal{O}(1)$ for flows with $Re\gg1$. The Carreau viscosity ratio is $c_0=(\mu_0^*-\mu_\infty^*)/\mu_\infty^*$ and $k=r^*\lambda^*\Omega^*\sqrt{\rho^*\Omega^*/\mu_\infty^*}$ is the dimensionless equivalent of the constant λ^*. Here

$$\hat{\mu}=\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\mathcal{L}_{\hat{\mu}}\right]^{1/2}, \tag{8h}$$

the higher order terms, $\mathcal{L}_{\hat{\mu}}$, that contribute to the generalised viscosity, are given in the Appendix A for completeness. We note that the expressions for B_n and k can be simplified when one considers the