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The rheometry of free surface flows ☆

P. Slatter *

ATC Williams, Melbourne, Australia

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ABSTRACT

Classically, open channel design for Newtonian turbulent flow is conducted using the pipe flow paradigm, incorporating the classical concept of the hydraulic radius. Although this approach has merit in turbulent flow, where inertial stresses dominate, it becomes problematic in laminar flow, where the tailings rheology dominates the flow behaviour. Arguably, the pipe flow paradigm for open channel flow has been taken as far as it is useful. The objective of this paper is to develop the fundamentals of sheet flow, and translate these into a sheet flow paradigm. Previous work on the pipe flow paradigm is reviewed, and the limits of its usefulness are exposed. A sheet flow approach is presented, developing a bulk shear rate which can be related directly to the tailings' rheology. This paradigm is then extended to a generalised laminar flow open channel flow model, incorporating the classical concept of the hydraulic radius. It is shown analytically that bulk shear rate is a unique function of wall shear stress for a given rheology and for all values of flow depth and slope. The significance of this is that in general the bulk shear rate is a unique function of the rheogram and the wall shear stress, and can be used for scale-up and design at any required slope and depth in laminar flow. Conversely, reversing the process provides the link between the pseudo shear diagram and the rheogram. This is validated using experimental results which show that the laminar data are collinear, the turbulent branches are succinct, and different for each slope, reminiscent of the pipe flow diagram. Experimental validation provides convincing evidence that the new sheet flow paradigm model for open channel flow can provide a competent basis for the analysis, flow behaviour prediction and design of open channel flow.

1. Introduction

Where sufficient elevational differences are available, free surface open channel transport is a viable option. Typically, the open channel design is conducted using the pipe flow paradigm, incorporating the classical concept of the hydraulic radius. Although this approach has merit in turbulent flow, where inertial stresses dominate and viscous stresses are negligible, it becomes problematic in laminar flow, where the tailings rheology dominates the flow behaviour. The fundamental point of departure of this work is that the pipe flow paradigm for open channel flow has been taken as far as it is useful. The objective of this paper is to develop the fundamentals of sheet flow, and translate these into a sheet flow paradigm. Previous work on the pipe flow paradigm is reviewed, and the limits of its usefulness are exposed. A sheet flow approach is presented, developing a bulk shear rate which can be related directly to the tailings' rheology. This paradigm is then extended to a generalised laminar flow open channel flow model, incorporating the classical concept of the hydraulic radius.

There are a variety of approaches available to measure the rheology of concentrated suspensions [1,2]. However it is generally not possible to obtain low shear rate data, especially in small tube diameters. The principal aim of the present work is to use laminar sheet flows to obtain flow curve data at lower shear rates, thereby supplementing other rheometry data.

The earliest work dealing with this topic is that of Astarita et al. [3]. However, they did not attempt to generate a flow curve over a range of shear rates or develop an expression for a bulk shear rate for rheological characterisation and scale up purposes. Instead, they found that the power law index values for polymer solutions obtained from inclined plane flow were similar to those found from rotational rheometry.

De Kee et al. [4] took this work further by extracting the values of the viscosity model parameters for power law, Bingham Plastic and Herschel–Bulkley models using the volumetric flow rate data from sheet flow experiments. However, the resulting values of the yield stress were only the fitted values that may or may not correspond to the true yield stress values if any.

Whilst all these workers built on earlier work [5], Uhlherr et al. [6], on the other hand, attempted to evaluate true (apparent) yield stress by locating the inclination angle demarcating the flow/no flow condition. The resulting values for a few silica suspensions

☆ Dedicated to Prof Ken Walters FRS on the occasion of his 80th Birthday.

* Tel.: +61 3 8587 0900; fax: +61 3 8587 0901.
E-mail address: pauls@atcwilliams.com.au

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and carbopol solutions were in line with those found from vane rheometry.

The objective of this paper is to directly use this sheet flow configuration to infer shear rate-shear stress behaviour of time independent fluids, akin to the approach of Rabinowitsch and Mooney [1,2,7] for tube flow.

2. Laminar tube flow of a general time-independent fluid

One of the primary objectives of rheometry is the establishment of the relationship between shear stress and shear rate. This is often referred to as a rheogram or a flow curve. For any time-independent fluid, this can be cast in the general form

$$\dot{\gamma} = f(\tau) \tag{1}$$

without reference to any particular rheological model.

For a cylindrical tube in the absence of wall-slip, it can be shown that the volumetric flowrate for steady laminar tube flow of a general time-independent fluid is given by [7]:

$$\frac{8V}{D} = \frac{32Q}{\pi D^3} = \frac{4}{\tau_0^3}\int_0^{\tau_0} \tau^2 \cdot f(\tau)d\tau \tag{2}$$

where $8V/D$ is the bulk shear rate, Q is the volumetric flow rate, τ_0 is the wall shear stress and $f(\tau)$ is the shear rate as defined by Eq. (1).

For Newtonian fluids, the bulk shear rate is equal to the shear rate at the wall. However, this is not the case for non-Newtonian fluids, and the two values are related to each other via the well-known Rabinowitsch and Mooney equation [7].

$$\dot{\gamma}_0 = \frac{8V}{D}\left(\frac{3n'+1}{4n'}\right) \tag{3}$$

where n' is defined as the local slope of the double logarithmic plot of wall shear stress, τ_0 vs. bulk shear rate, $8V/D$:

$$n' = \frac{d\ln\tau_0}{d\ln\frac{8V}{D}}. \tag{4}$$

The aim of this present work is to develop relationships similar to the above for sheet flow.

3. Sheet flow analysis

Consider the steady incompressible and laminar flow of a time-independent fluid on an inclined plane, as shown in Fig. 1.

The free surface of the sheet is assumed to be smooth and free from ripples. Furthermore, the sheet thickness, H is assumed to be uniform and $\ll W$, the width of the plate in the z-direction. Under these conditions, there is only one non-zero shear stress component τ_{xy} given by

$$\tau_{xy} = \rho g y \sin\alpha \tag{5}$$

The shear stress varies linearly in the y-direction from being zero at $y = D$ at the free surface to its maximum value τ_0 at the wall given by:

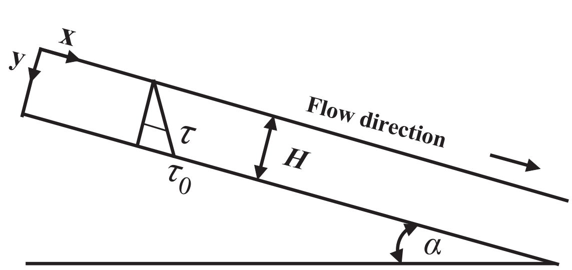

Fig. 1. Sheet flow shear stress distribution.

$$\tau_0 = \rho g H \sin\alpha \tag{6}$$

From Eqs. (5) and (6), it can be deduced that:

$$\frac{\tau_{xy}}{\tau_0} = \frac{y}{H} \tag{7}$$

In this case, the only non-zero component of the velocity is u_x which is a function of y only, i.e. $u_x(y)$.

The volumetric flow rate per unit width Q, of the liquid in the sheet is given by

$$Q = \int_0^H u_x dy \tag{8}$$

Integrating by parts yields

$$Q = [u_x y]_0^H - \int_0^H \left(y\frac{du_x}{dy}\right)dy \tag{9}$$

Assuming the no-slip condition at the wall, i.e. $u_x = 0$ at $y = H$, Eq. (9) reduces to

$$Q = \int_0^H y\left(-\frac{du_x}{dy}\right)dy \tag{10}$$

For a time-independent fluid, we can describe the relationship between shear rate, $\dot{\gamma}$ and shear stress, τ as:

$$-\frac{du_x}{dy} = \dot{\gamma} = f(\tau) \tag{11}$$

Substituting Eq. (11) and changing the variable of integration from y to τ, Eq. (10) can be rewritten as

$$Q = \frac{H^2}{\tau_0^2}\int_0^{\tau_0} \tau f(\tau)d\tau \tag{12}$$

For a Newtonian fluid, the constitutive relationship is

$$f(\tau) = \frac{\tau}{\mu} \tag{13}$$

Substitution in Eq. (12) leads directly to the well-known result for Newtonian laminar sheet flow [5]

$$Q = \frac{H^2\tau_0}{3\mu} \tag{14}$$

In introducing the average velocity of flow V as Q/H, Eq. (14), yields

$$V = \frac{Q}{H} = \frac{H\tau_0}{3\mu} \tag{15}$$

Combining Eqs. (13) and (15) and expressing this in terms of the wall shear stress and wall shear rate leads to

$$\tau_0 = \mu\frac{3V}{H} = \mu\dot{\gamma}_0 \tag{16}$$

It follows from Eq. (16) that the term $3V/H$ is the shear rate at the wall for a Newtonian fluid, i.e.

$$\dot{\gamma}_{0_{Newt}} = \frac{3V}{H} \tag{17}$$

It is further proposed that in general, $3V/H$ is the bulk shear rate for sheet flow. Hence, Eq. (12) can be rewritten as

$$\frac{3V}{H} = \frac{3}{\tau_0^2}\int_0^{\tau_0} \tau \cdot f(\tau)d\tau \tag{18}$$

As with tube flow, the dilemma now arises as to how the true shear rate at the wall can be expressed in terms of the bulk shear rate. We now develop a relationship similar to the Rabinowitsch–Mooney equation [7] for sheet flow [8].