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Stability of the boundary layer on a rotating disk for power-law fluids

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ABSTRACT

The stability of the flow due to a rotating disk is considered for non-Newtonian fluids, specifically shear-thinning fluids that satisfy the power-law (Ostwald-de Waele) relationship. In this case the basic flow is not an exact solution of the Navier–Stokes equations, however, in the limit of large Reynolds number the flow inside the three-dimensional boundary layer can be determined via a similarity solution. An asymptotic analysis is presented in the limit of large Reynolds number. It is shown that the stationary spiral instabilities observed experimentally in the Newtonian case can be described for shear-thinning fluids by a linear stability analysis. Predictions for the wavenumber and wave angle of the disturbances suggest that shear-thinning fluids may have a stabilising effect on the flow.

1. Introduction

The stability of the boundary layer on a rotating disk due to the flow of a Newtonian fluid is a classical problem that has attracted a great deal of attention from numerous authors over many decades. The first theoretical investigation of this problem was performed by von Kármán [1]. The steady flow induced by the rotation of an infinite plane with uniform angular velocity is an exact solution of the Navier–Stokes equations. The flow is characterised by the lack of a radial pressure gradient near to the disk to balance the centrifugal forces, so the fluid spirals outwards. The disk acts as a centrifugal fan, the fluid emanating from the disk being replaced by an axial flow directed back towards the surface of the disk.

Batchelor [2] showed that this type of flow is in fact just a limiting case of a whole number of flows with similarity solutions in which both the infinite plane and the fluid at infinity rotate with differing angular velocities. The corresponding limiting case when the infinite plane is stationary and the fluid at infinity rotates at a constant angular velocity was first described by Bödewadt [3].

The stability of the von Kármán flow was first investigated by Gregory et al. [4]. They observed spiral modes of instability in the form of co-rotating vortices, measuring the angle between the normal to the radius vector and the tangent to the vortices to be $\phi \approx 13^{\circ}$. Gregory et al. [4] showed that these experimental observations were in excellent agreement with their own predictions obtained from a linear stability analysis. Hall [5] extended this work taking into account the viscous effects, showing that an additional stationary short-wavelength mode exists which has its structure fixed by a balance between viscous and Coriolis forces.

There have been several numerical studies of the stability of the von Kármán boundary layer. Examples include that of Malik [6], Lingwood [7]. Both studies used a parallel-flow approximation for the basic flow. Malik [6] considered convective instability and presented results for stationary vortices, finding that for a large Reynolds number $\phi \approx 13^{\circ}$ for inviscid neutrally stable modes. Lingwood [7] extended these results by considering Ekman and Bödewadt flows. She also investigated the absolute instability of these flows, showing that the von Kármán boundary layer is locally absolutely unstable for Reynolds number above a critical value. Subsequently, Davies and Carpenter [8] considered the global behaviour of the absolute instability of the rotating-disk boundary layer. By direct numerical simulations of the linearised governing equations they were able to show that the local absolute instability does not produce a linear global instability. Suggesting that, instead, convective-type behaviour dominates, even within the region of local absolute instability.

Considerably less attention has been given to the problem of the boundary layer flow due to a rotating disk when considering a non-Newtonian fluid. Mitschka [9] extended the von Kármán solution to fluids that adhere to the power-law relationship. In this case the basic flow is not an exact solution of the Navier–Stokes equations and a boundary layer approximation is required. Both Mitschka and Ulbrecht [10], Andersson et al. [11] present

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numerical solutions for the basic flow for shear-thickening and shear-thinning fluids. However, both sets of authors overlooked the importance of matching this boundary-layer flow with the outer flow. Denier and Hewitt [12] addressed this problem and presented corrected similarity solutions of the boundary-layer equations. This involved a comprehensive knowledge of the far-field behaviour. Their analysis revealed different situations for shear-thinning and shear-thickening fluids. For shear-thickening fluids the boundary-layer solution is complicated by a region of zero viscosity away from the boundary. For the more common shear-thinning fluids, beyond a critical level of shear-thinning, the basic flow solution grows in the far field, so it cannot be matched to an external flow. For more details of these cases the reader is referred to Denier and Hewitt [12].

Thus, in the current paper we restrict our attention to moderate levels of shear-thinning, where the boundary-layer solution may be matched to an outer flow (although this will not be in similarity form). In this case we can use a boundary-layer similarity solution to give an analytic description of the stability of the three-dimensional flow for large Reynolds numbers. This only requires knowledge of the boundary layer since this is where the vortices are confined.

Specifically, we look to extend the previous works concerning convective instability of Newtonian flows to include the additional viscous effects of a power-law fluid. The current study will follow the approach of Hall [5] to investigate the so called stationary "inviscid instabilities" with vortices occurring at the location of an inflection point of the effective velocity profile.

2. Formulation

Consider the flow of a steady incompressible non-Newtonian fluid due to a rotating disk located at $z = 0$. The disk rotates about the z-axis with angular velocity Ω. Working in a reference frame that rotates with the disk, the dimensionless continuity and Navier–Stokes equations are expressed as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2[(\hat{\boldsymbol{z}} \times \boldsymbol{u}) - r\hat{\boldsymbol{r}}] = -\nabla p + \frac{1}{Re}\nabla \cdot \boldsymbol{\tau}. \tag{2}$$

Here $\boldsymbol{u} = (u, v, w)$ are the velocity components in cylindrical polar coordinates (r, θ, z) where r and z have been made dimensionless with respect to some reference length l and $(\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\boldsymbol{z}})$ are the corresponding unit vectors in the respective coordinate directions. The velocities and pressure have been non-dimensionalised by Ωl and $\rho\Omega^2 l^2$ respectively, the fluid density is ρ and p is the fluid pressure. The stress tensor $\boldsymbol{\tau}$ for incompressible non-Newtonian fluids is given by the generalised Newtonian model

$$\boldsymbol{\tau} = \mu\dot{\boldsymbol{\gamma}} \quad \text{with} \quad \mu = \mu(\dot{\gamma}), \tag{3}$$

where $\dot{\boldsymbol{\gamma}} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T$ is the rate of strain tensor and $\mu(\dot{\gamma})$ is the non-Newtonian viscosity. The magnitude of the rate of strain tensor is

$$\dot{\gamma} = \sqrt{\frac{\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}}{2}}. \tag{4}$$

The governing relationship for $\mu(\dot{\gamma})$ when considering a power-law fluid is

$$\mu(\dot{\gamma}) = m(\dot{\gamma})^{n-1}, \tag{5}$$

where m is known as the consistency coefficient and n the power-law index, with $n > 1$, $n < 1$ corresponding to shear-thickening and shear-thinning fluids, respectively. The modified non-Newtonian Reynolds number is defined as $Re = \rho\Omega^{2-n}l^2/m$.

In the Newtonian case an exact solution of the Navier–Stokes equations exists, as was first determined by von Kármán [1]. Due to the relative complexity of the modified stress tensor no such solution exists when considering the flow of a power-law fluid. However, in the limit of large Reynolds number progress can be made as the leading order boundary-layer equations admit a similarity solution analogous to the exact solution obtained in the Newtonian problem.

As noted by Denier and Hewitt [12] the boundary-layer equations at lowest order are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_B) + \frac{\partial w_B}{\partial z} = 0, \tag{6a}$$

$$u_B\frac{\partial u_B}{\partial r} + w_B\frac{\partial u_B}{\partial z} - \frac{(v_B + r)^2}{r} = \frac{1}{Re}\frac{\partial}{\partial z}\left(\mu_B\frac{\partial u_B}{\partial z}\right), \tag{6b}$$

$$u_B\frac{\partial v_B}{\partial r} + w_B\frac{\partial v_B}{\partial z} + \frac{u_B v_B}{r} + 2u_B = \frac{1}{Re}\frac{\partial}{\partial z}\left(\mu_B\frac{\partial v_B}{\partial z}\right), \tag{6c}$$

where

$$\mu_B = \left[\left(\frac{\partial u_B}{\partial z}\right)^2 + \left(\frac{\partial v_B}{\partial z}\right)^2\right]^{(n-1)/2}. \tag{6d}$$

To solve for the basic flow inside the boundary layer Mitschka [9] introduced a similarity solution of the form

$$\boldsymbol{u_B} = [r\bar{u}(\eta), r\bar{v}(\eta), r^{(n-1)/(n+1)}Re^{-1/(n+1)}\bar{w}(\eta)], \tag{7}$$

where the similarity variable η is given by

$$\eta = r^{(1-n)/(n+1)}Re^{1/(n+1)}z. \tag{8}$$

The dimensionless functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ are determined, after substitution of (7) into (6a)–(6c) and (6d) by

$$2\bar{u} + \frac{1-n}{n+1}\eta\bar{u}' + \bar{w}' = 0, \tag{9a}$$

$$\bar{u}^2 - (\bar{v}+1)^2 + \left(\bar{w} + \frac{1-n}{n+1}\eta\bar{u}\right)\bar{u}' - [(\bar{u}'^2 + \bar{v}'^2)^{(n-1)/2}\bar{u}']' = 0, \tag{9b}$$

$$2\bar{u}(\bar{v}+1) + \left(\bar{w} + \frac{1-n}{n+1}\eta\bar{u}\right)\bar{v}' - [(\bar{u}'^2 + \bar{v}'^2)^{(n-1)/2}\bar{v}']' = 0, \tag{9c}$$

where the primes denote differentiation with respect to η. The appropriate boundary conditions are

$$\bar{u} = \bar{v} = \bar{w} = 0 \quad \text{at} \quad \eta = 0, \tag{10a}$$

$$\bar{u} \to 0, \quad \bar{v} \to -1 \quad \text{as} \quad \eta \to \infty. \tag{10b}$$

Denier and Hewitt [12] have shown that bounded solutions to 9a, 9b and 9c subject to (10a) and (10b) exist only in the shear-thinning case for $n > \frac{1}{2}$. In the shear-thickening case they have shown that solutions become non-differentiable at some critical location η_c, and although it transpires that this singularity can be regularised entirely within the context of the power-law model, we will not consider such flows here. Thus in this study we will consider flows with power-law index in the range $\frac{1}{2} < n \leqslant 1$. They have also shown that for $\frac{1}{2} < n < 1$ to ensure the correct algebraic decay in the numerical solutions one must apply the Robin condition

$$(\bar{u}', \bar{v}') = \frac{n}{\eta(n-1)}(\bar{u}, \bar{v}) \quad \text{as} \quad \eta \to \infty, \tag{11}$$

at some suitably large value of $\eta = \eta_\infty \gg 1$. In the Newtonian case this relationship becomes singular, this is due to the fact that when $n = 1$ the functions $\bar{u}$ and $\bar{v}$ decay exponentially. Cochran [13] showed that in this case

$$(\bar{u}', \bar{v}') = \bar{w}_\infty(\bar{u}, \bar{v}) \quad \text{as} \quad \eta \to \infty, \tag{12}$$

where $w_\infty = -2\int_0^\infty \bar{u}\,\mathrm{d}\eta$.

Numerical solutions of 9a, 9b and 9c subject to (10a) and (10b) are presented in Table 1 and Fig. 1. These results were obtained using a fourth-order Runge–Kutta quadrature routine twinned