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Short Communication

# The periodic solution of Stokes' second problem for viscoelastic fluids as characterized by a fractional constitutive equation

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ABSTRACT

Stokes' second problem is about the steady-state oscillatory flow of a viscous fluid due to an oscillating plate. We consider Stokes' second problem for a class of viscoelastic fluids that are characterized by a fractional constitutive equation. The exact analytical solution as parametrized by the order of the fractional derivative is obtained. We provide detailed analyses and discussions for effects of the model parameters on the wave length and the amplitude in the flow field. We show that, as the order varies from 0 to 1, the flow displays a transition from elastic to viscous behavior. Finally, we consider the case of the constitutive equation for a fractional element or a spring-pot in series with a dashpot.



## 1. Introduction

Stokes' second problem is a benchmark problem in fluid mechanics. It describes the steady-state oscillatory flow in a semi-infinite flow domain arisen from an oscillating infinite flat plate that undergoes sinusoidal oscillations parallel to itself. Only the steady periodic solution, after the starting transients have died, will be considered; thus there are no initial conditions to satisfy. In Stokes' second problem, the flow velocity $\mathbf{v} = u(y,t)\mathbf{i}$, where $\mathbf{i}$ is the unit vector along the $x$-axis of the Cartesian coordinate system, satisfies the diffusion equation [1,2]

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2}, \; y > 0, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = U\cos(\omega t), \; u(+\infty, t) = 0, \tag{2}$$

where $v$ is the kinematic viscosity, and $U$ and $\omega$ are constants. Thus the solution of Stokes' second problem is

$$u(y,t) = U\exp\left(-y\sqrt{\frac{\omega}{2v}}\right)\cos\left(\omega t - y\sqrt{\frac{\omega}{2v}}\right), \tag{3}$$

which is periodic with respect to $t$.

In this work, we consider Stokes' second problem for a class of viscoelastic fluids as characterized by a fractional constitutive equation. In recent years, the fractional calculus has been introduced to effectively formulate the constitutive relations of viscoelastic fluids [3–10].

Let $f(t)$ be piecewise continuous on $(-\infty, +\infty)$, then the Riemann–Liouville fractional integral of $f(t)$ of order $\alpha$ is defined as [5,6,11–14]

$$\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}t^{-\alpha}}f(t) = \int_{-\infty}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds, \quad \alpha > 0, \tag{4}$$

where $\Gamma(\cdot)$ is Euler's gamma function. In this work, we take the lower limit of the integral in Eq. (4) to be negative infinity in order to adapt it to the steady-state problem, where no initial conditions are to be satisfied.

The Caputo fractional derivative of $f(t)$ of order $\alpha$ is defined as [5,6,11–14]

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t) = \frac{\mathrm{d}^{-(n-\alpha)}}{\mathrm{d}t^{-(n-\alpha)}}f^{(n)}(t), \quad 0 \leqslant n-1 < \alpha < n, \; n \in \mathbb{N}^{+}. \tag{5}$$

We assume that the considered viscoelastic fluid complies with the fractional constitutive relation for the shear stress $\tau$ and the shear strain $\epsilon$ [3–6,8,15]

$$\tau = G\lambda^{\beta}\frac{\mathrm{d}^{\beta}\epsilon}{\mathrm{d}t^{\beta}} = G\lambda^{\beta}\frac{\mathrm{d}^{\beta-1}\dot{\epsilon}}{\mathrm{d}t^{\beta-1}}, \; 0 < \beta < 1, \tag{6}$$

where $G$ is the shear modulus and $\lambda$ is the relaxation time. A fractional calculus element whose constitutive law satisfies Eq. (6) is said to be a spring-pot [3].

In Section 3, we will consider a constitutive equation where a fractional element connects in series with a dashpot.

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## 2. Governing equation and solution

Suppose that an incompressible viscoelastic fluid occupies the space over a flat plate of infinite extent situated in the $(x, z)$ plane. The plate oscillates in its own plane with the velocity $U\cos(\omega t)\mathbf{i}$, where the amplitude $U$ is assumed to be small. Owing to the viscosity, the fluid above the plate also moves, its velocity being of the form $\mathbf{v} = \mathbf{v}(y, t) = u(y, t)\mathbf{i}$. The schematic of the problem under consideration is shown in Fig. 1. We consider the case of steady-state flow, i.e. the fully developed flow such that for each specified $y > 0$, the velocity $u(y, t)$ is periodic in time $t$.

In the absence of body forces and a pressure gradient, the balance equations of mass and momentum governing the flow of an incompressible fluid are

$$\nabla \cdot \mathbf{v} = 0,\ \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \mathbf{S}, \tag{7}$$

where $\mathbf{v} = (u, v, w)$ is the velocity, $\rho$ is the density of the fluid, $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ is the material time derivative, and $\mathbf{S}$ is the extra stress tensor.

For our particular Stokes' second problem, $\mathbf{v} = u(y, t)\mathbf{i}$ and hence the motion equation is reduced to

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial S_{12}}{\partial y}, \tag{8}$$

where $S_{12}$ is the shear stress. Substituting the fractional constitutive equation

$$S_{12} = G\lambda^{\beta} \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} \frac{\partial u}{\partial y},\ \ 0 < \beta < 1, \tag{9}$$

we obtain the governing equation

$$\frac{\partial u}{\partial t} = \nu(\beta) \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} \frac{\partial^2 u}{\partial y^2},\ y > 0,\ 0 < \beta < 1, \tag{10}$$

where

$$\nu(\beta) = G\lambda^{\beta}/\rho, \tag{11}$$

and the dimension is $[\nu(\beta)] = \frac{L^2}{T^{2-\beta}}$. The velocity $u = u(y, t)$ satisfies the boundary conditions

$$u(0, t) = U\cos(\omega t),\ U > 0,\ \omega > 0, \tag{12}$$

$$u(+\infty, t) = 0. \tag{13}$$

We note that there are no initial conditions since we consider the steady-state oscillating flow.

First, we look for the solution of complex values of Eq. (10) satisfying the boundary conditions

$$u(0, t) = Ue^{i\omega t},\ u(+\infty, t) = 0. \tag{14}$$

Then we take the real part of the solution of complex values to obtain the solution of problem (10)–(13).

We assume that the solution in terms of complex values permits the separation of variable as

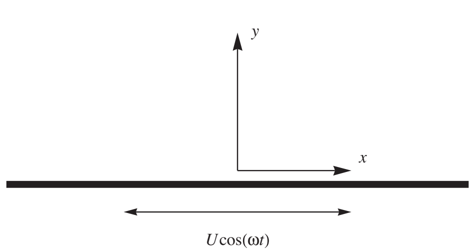


**Fig. 1.** Schematic diagram of Stokes' second problem.

$$u(y, t) = f(y)e^{i\omega t}, \tag{15}$$

where $f(y)$ satisfies

$$f(0) = U,\ f(+\infty) = 0, \tag{16}$$

due to the boundary condition (14).

Substituting Eq. (15) into Eq. (10), we readily deduce that

$$f(y)e^{i\omega t}(i\omega) = \nu(\beta)f''(y) \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} e^{i\omega t}. \tag{17}$$

The fractional integral on the right hand side is calculated to be

$$\begin{aligned} \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} e^{i\omega t} &= \int_{-\infty}^{t} \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} e^{i\omega \tau} d\tau = e^{i\omega t} \int_{0}^{+\infty} \frac{s^{-\beta}}{\Gamma(1-\beta)} e^{-i\omega s} ds \\ &= e^{i\omega t}(i\omega)^{\beta-1}. \end{aligned} \tag{18}$$

Inserting Eq. (18) into Eq. (17) yields

$$f''(y) = (\nu(\beta))^{-1}(i\omega)^{2-\beta} f(y). \tag{19}$$

Therefore the general solution of Eq. (19) is

$$\begin{aligned} f(y) = &A\exp\left(-(\nu(\beta))^{-1/2}(i\omega)^{1-\frac{\beta}{2}}y\right) + B \\ &\times \exp\left((\nu(\beta))^{-1/2}(i\omega)^{1-\frac{\beta}{2}}y\right). \end{aligned} \tag{20}$$

From Eq. (16), we deduce that $B = 0$ and $A = U$.

Hence the solution of problem (10)–(13) is obtained by calculating the real part as

$$\begin{aligned} u(y, t) &= \mathrm{Re}[U\exp(i\omega t - (\nu(\beta))^{-1/2}(i\omega)^{(2-\beta)/2}y)] \\ &= U\exp(-\eta_1)\cos(\omega t - \eta_2), \end{aligned} \tag{21}$$

where

$$\eta_1 = y\sqrt{\frac{\omega^{2-\beta}}{\nu(\beta)}} \sin\frac{\pi\beta}{4},\ \ \eta_2 = y\sqrt{\frac{\omega^{2-\beta}}{\nu(\beta)}} \cos\frac{\pi\beta}{4}. \tag{22}$$

We observe that, in the solution (21), there are three nondimensional variables: $u/U, \omega t$ and $y\sqrt{\omega^{2-\beta}/\nu(\beta)}$. We also note that what we do is a Fourier transform of the governing equation with respect to $t$.

## 3. Results and discussion

Eq. (21) represents harmonic vibrations for each specified $y > 0$. The flow field has the same oscillating frequency $\omega$ as the plate, and its amplitude is $U\exp(-\eta_1)$. As the value of $y$ increases the amplitude decreases exponentially. The phase difference of the flow velocity at a point with $y$ coordinate above the plate and the plate vibration is $\eta_2$. The wave length of the fluid vibration is

$$l = 2\pi\sqrt{\frac{G\lambda^{\beta}}{\rho\omega^{2-\beta}}} \sec\frac{\pi\beta}{4}, \tag{23}$$

which is obtained by setting $\eta_2 = 2\pi$ in Eq. (22) and using Eq. (11). Two flow layers with the distance $l$ have the same phases. The wave length $l$ monotonically decreases as the oscillating frequency $\omega$ increases. In Fig. 2, we plot the curves of the wave length $l$ versus the oscillating frequency $\omega$ for $G/\rho = 1, \lambda = 1$ and for $\beta$ = 0, 0.5 and 1, respectively.

From the derivative

$$\frac{dl}{d\beta} = \pi\sqrt{\frac{G\lambda^{\beta}}{\rho\omega^{2-\beta}}} \sec\left(\frac{\pi\beta}{4}\right)\left(\frac{\pi}{2}\tan\left(\frac{\pi\beta}{4}\right) + \ln(\lambda\omega)\right), \tag{24}$$

we observe that, if $\lambda$ and $\omega$ satisfy the inequality $\ln(\lambda\omega) \leqslant -\pi/2$, then the wave length $l$ monotonically decreases as $\beta$ increases from 0 to 1; if $-\pi/2 < \ln(\lambda\omega) < 0$, then the wave length $l$ has a minimum