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Transient electro-osmotic flow of Oldroyd-B fluids in a straight pipe of circular cross section

Moli Zhao [a], Shaowei Wang [a,b,*], Shoushui Wei [c]

[a] Geotechnical and Structural Engineering Research Center, Shandong University, Jinan 250061, Shandong, PR China
[b] Department of Engineering Mechanics, School of Civil Engineering, Shandong University, Jinan 250061, PR China
[c] School of Control Science and Engineering, Shandong University, Jinan 250061, Shandong, PR China

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ABSTRACT

The transient electro-osmotic flow of viscoelastic fluids in a narrow capillary tube is examined. With the help of integral transform method, analytical expressions are derived for the electric potential and transient velocity profile by solving the linearized Poisson–Boltzmann equation and the Navier–Stokes equation. It is shown that the distribution and establishment of the velocity consists of two parts, the steady part and the unsteady one. The results of classical fluid, i.e., Newtonian fluid and those of Maxwell fluid and the second grade fluid can be obtained as the special cases of the results in present study. The effects of relaxation time and retardation time on the velocity profiles are analyzed numerically. It is pointed out that the electro-osmotic flow of viscoelastic fluids is more difficult to achieve the steady state.

1. Introduction

With the development of microfluidic devices and their applications in microelectromechanical system and microbiological sensors [1–3], the research field of electro-osmosis (EO) has become very attractive. Recently, some researchers [4–6] pointed out that the micelle structure of polymer electrolyte membranes (PEMs) might consist of only cylindrical nano-channels, which facilitate water and proton transport, rather than large water pore clusters connected by smaller nano-channels as in Gierke's model. This raises the problem that how to model the fluids electro-osmotic flow in a straight pipe of circular cross section.

Most of the theoretical researches on electro-osmotic flow are limited to the fully developed steady-state flow [7–11]. An electro-osmotic flow problem in an infinite cylindrical pore with a uniform surface charge density has been studied analytically by Berg and Ladipo [12], the results revealed the distribution of the electric potential and the counter-ions (protons), the velocity profile of the water flow and its associated total flux, as well as the protonic current, conductivity and water drag. Chang [13] presented a theoretical study on the transient electro-osmotic flow through a cylindrical microcapillary containing a salt-free medium for both constant surface charge density and constant surface potential, the exact solutions for the electric potential distribution and the transient electro-osmotic flow velocity are derived by solving the nonlinear Poisson–Boltzmann equation and the Navier–Stokes equation. With the application of a stepwise voltage, Mishchuk and González-Caballero studied a theoretical model of electro-osmotic flow in a wide capillary [14], both periodical and aperiodical flow regimes were studied with arbitrary pulse/pulse or pulse/pause durations and amplitudes.

On the other hand, microfluidic devices are usually used to analyze biofluids, which are often solutions of long chain molecules and their behavior are very different from that of Newtonian fluid, such as memory effects, normal stress effect, and yield stress. These fluids cannot be treated as Newtonian fluids. Many researchers have recently focused on non-Newtonian behavior of biofluids in electrokinetically driven microflows. The first research of non-Newtonian effects to electro-osmotic flow was done by Das and Chakraborty [15] and Chakraborty [16], in their studies, the biofluids were treated as power-law fluids, and the analytical solution, describing the transport characteristics of a non-Newtonian fluid flow in a rectangular microchannel, was obtained under the sole influence of electrokinetic effects. For the same non-Newtonian fluid model, Zhao and Yang [17,18] obtained the general Smoluchowski velocity for electro-osmosis over a surface with arbitrary zeta potentials. Park and Lee [19] derived a semi-analytical expression for the

* Corresponding author at: Geotechnical and Structural Engineering Research Center, Shandong University, Jinan 250061, Shandong, PR China. Tel.: +86 531 88399320.
E-mail address: shaoweiwang@sdu.edu.cn (S. Wang).

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Helmholtz–Smoluchowski velocity under pure electroosmosis conditions for the full Phan–Thien–Tanner (PTT) constitutive equation, and they used a finite volume method to calculate numerically the flow of the full PTT model in a rectangular duct under the action of electroosmosis and a pressure gradient [20]. With the help of Fourier transform, Bandopadhyay and Chakraborty [21] investigated the dynamical interplay between interfacial electrokinetics and a combined dissipative and elastic behavior of ow in narrow connements. Recently, they addressed the implications of finite sizes of the ionic species on electroosmotic transport through in narrow confinements in the case of a counterion-only solution, and pointed out that the electroosmotic mobility is dependent on both the size of the channel and the size of the ions [22].

In present study, the non-Newtonian behavior of biofluids is modelled by the Oldroyd-B constitutive equation. The purpose of this paper is to present the analytical solution of unsteady electro-osmotic flow of Oldroyd-B fluids in a cylindrical capillary.

2. Governing equations

2.1. Constitutive equation of Oldroyd-B fluid

The continuity equation for an incompressible fluid is

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

and the general Cauchy momentum equation

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \nabla\cdot\boldsymbol{\sigma} + \mathbf{F}, \tag{2}$$

here $\mathbf{V}$ is the velocity vector, ρ is the fluid density, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{F}$ is the extemal body force vector, and ∇ is the gradient operator.

The Cauchy stress tensor $\boldsymbol{\sigma}$ for Oldroyd-B fluid is

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}, \quad \left(1+\lambda_1\frac{D}{Dt}\right)\boldsymbol{\tau} = \boldsymbol{\mu}\left(1+\lambda_2\frac{D}{Dt}\right)\mathbf{A_1}, \tag{3}$$

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^T, \tag{4}$$

$$\frac{D\boldsymbol{\tau}}{Dt} = \frac{\partial\boldsymbol{\tau}}{\partial t} + (\mathbf{V}\cdot\nabla)\boldsymbol{\tau} - (\nabla\mathbf{V})\cdot\boldsymbol{\tau} - \tau\cdot(\nabla\mathbf{V})^T, \tag{5}$$

where $\boldsymbol{\tau}$ is extra stress tensor, $\mathbf{I}$ is the unit tensor, p is the pressure and the superscript T denotes the tensor transpose, λ_1 and λ_2 are relaxation time and retardation time of the Oldroyd-B fluid, respectively.

2.2. Mathematical model of the flow

Consider the electro-osmotic flow of Oldroyd-B fluid of dielectric constant ε, at rest at time $t \leqslant 0$, contained in a straight pipe of circular cross section and radius R. It is assumed that the pipe wall is uniformly charged with a zeta potential, ψ_w. When an external electric field E_0 is imposed along the axial direction, the fluid in the pipe sets in motion due to electro-osmosis.

All quantities are referred to cylindrical polar coordinates (r, θ, z), where r is measured from the axis of the pipe and z along it. If we assume a velocity distribution of the form

$$(0, 0, u(r,t)), \quad 0 \leqslant r \leqslant R, \quad t > 0, \tag{6}$$

the initial condition is given by

$$u(r,0) = 0, \quad 0 \leqslant r \leqslant R, \tag{7}$$

and the equation of continuity (1) is satisfied automatically.

According to the theory of electrostatics, the net charge density ρ_e is expressed by a potential distribution ψ, which is given by the Poison equation,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = -\frac{\rho_e}{\varepsilon}. \tag{8}$$

The boundary condition is that the zeta potential ψ_w is given on the wall of the pipe,

$$\psi(R,\theta) = \psi_w, \quad \left.\frac{\partial\psi}{\partial r}\right|_{r=0} = 0. \tag{9}$$

In present research, we assume that the charge distribution in the Debye layer is not affected by time, i.e., the wall of the pipe has constant electric potential E_0. Then the relevant equation of motion reduces to

$$\rho\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) - \rho_e E_0, \tag{10}$$

which has the following initial and boundary conditions

$$u(r,0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{11}$$

$$u(r,t) = 0, \quad r = R. \tag{12}$$

3. Exact solution for the model

Neglecting all non-electrostatic interactions between the ions including the ionic finite size, i.e., here we assume that the ions are point sized, for small values of electrical potential ψ of the electrical double layer (EDL), the Debye–Hückel approximation can be used successfully, which means physically that the electrical potential is small compared with the thermal energy of the charged species. So we have the linearized charge density

$$\rho_e = -\frac{2z_v^2 e^2 n_0 \psi}{k_B T}, \tag{13}$$

where z_v is the valence of ions, e is the fundamental charge, k_B is the Boltzmann constant, T is the absolute temperature.

With the help of the Debye–Hückel approximation [23,24], Eq. (8) can be linearized to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \kappa^2\psi. \tag{14}$$

Then the equation of motion (10) becomes

$$\rho\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) - \kappa^2\varepsilon\psi E_0, \tag{15}$$

here $\kappa = (2z_v^2 e^2 n_0/\varepsilon k_B T)^{1/2}$ is the Debye–Hückel parameter and κ^{-1} means the thickness of EDL.

Choosing the cylindrical coordinate (r, θ, z), the constitutive equation for Oldroyd-B fluid can be expressed as

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\tau_{rz} = \left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}. \tag{16}$$

Eliminating τ_{rz} from (15) and (16) yields

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\rho\frac{\partial u}{\partial t} + \kappa^2\varepsilon\psi E_0\right] = \mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \tag{17}$$

Introducing the below listed non-dimensional parameters,

$$\psi^* = \frac{\psi}{\psi_w}, \quad u^* = \frac{u}{u_s}, \quad r^* = \frac{r}{R}, \quad t^* = \frac{\mu}{R^2\rho}t, \quad u_s = -\frac{\varepsilon\psi_w E_0}{\mu}, \tag{18}$$