European Journal of Control 22 (2015) 23–29

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

Ensemble controllability and discrimination of perturbed bilinear control systems on connected, simple, compact Lie groups

Mohamed Belhadj [a,*], Julien Salomon [b], Gabriel Turinici [b,c]

[a] Département de Mathématiques, Institut Supérieur des Mathématiques Appliquées et d'Informatique de Kairouan, Av. Assad Ibn El Fourat, 3100 Kairouan, Tunisia
[b] CEREMADE, Université Paris Dauphine, Pl. du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France
[c] Institut Universitaire de France, France

ARTICLE INFO

Article history:
Received 2 June 2014
Received in revised form
11 December 2014
Accepted 21 December 2014
Recommended by F. Mazenc
Available online 2 February 2015

Keywords:
Quantum control
Lie group controllability
Bilinear system
Perturbations

ABSTRACT

The controllability of bilinear systems is well understood for finite dimensional isolated systems where the control can be implemented exactly. However when perturbations are present some interesting theoretical questions are raised. We consider in this paper a control system whose control cannot be implemented exactly but is shifted by a time independent constant in a discrete list of possibilities. We prove under general hypothesis that the collection of possible systems (one for each possible perturbation) is simultaneously controllable with a common control. The result is extended to the situations where the perturbations are constant over a common, long enough, time frame. We apply the result to the controllability of quantum systems. Furthermore, some examples and a convergence result are presented for the situation where an infinite number of perturbations occurs. In addition, the techniques invoked in the proof allow us to obtain generic necessary and sufficient conditions for ensemble controllability.

1. Introduction

The fundamental importance of addressing the controllability of bilinear systems has long been recognized in engineering control applications (see [40,21,31,24,34,12,1,2,48]). Among recent applications one may cite the field of quantum control with optical or magnetic external fields (see [35,18,19,34,46,36,47,48,14,7,8,15]).

Although the controllability is well understood when the system is of finite dimension, isolated and the control can be implemented exactly, new theoretical and numerical questions are raised when perturbations are present.

The question that is addressed in this paper is related to the simultaneous controllability of bilinear systems.

Consider a collection of control systems with states X_k, $k=1,\dots,K$ in Lie groups G_k evolving according to $dX_k(t)/dt=(A_k+u(t)B_k)X_k$. Simultaneous controllability (also called "ensemble controllability") is the question of whether all states X_k can be controlled with the same control $u(t)$. We will use the terms "simultaneous controllability" and "ensemble controllability" interchangeably.

Problems of simultaneous control of a finite collection of systems have been addressed recently in applications related to quantum control [50,28,49,33,37,25–27,42,44,3,30]. In such circumstances, the system is a collection of molecules or atoms or spin systems and the control is a magnetic field (in NMR) or a laser. The assessment of whether a single control pulse can drive independent (i.e., distinct) quantum systems to their respective target states was addressed theoretically in [50] for general A_k, B_k and applied to the optimal dynamic discrimination of separate quantum systems in [28]. The particular case of identical molecules with $A_k=A$ (constant) and $B_k=\xi_k B, \xi_k\in\mathbb{R}$, $G=\mathrm{SU}(N)$ was treated in [49,33] where, under some technical assumptions on A and B, it is proved that all members of an ensemble of randomly oriented molecules subjected to a single ultra-fast laser control pulse can be simultaneously controlled. An independent work [3] treats the circumstance when $A_k=\epsilon_k A$, $|\epsilon_j|\neq|\epsilon_\ell|$ for any $j\neq\ell$, $G=\mathrm{SU}(N)$ and $B_k=B$ (constant) and was used to show controllability for ensembles N-level quantum systems having different Larmor dispersion. This last result generalizes the findings of [25] for ensembles of spin 1/2 systems.

The infinite dimensional version (an infinite number of systems $A_\epsilon=\epsilon A$ with ϵ taking arbitrary values in an interval $]\epsilon_*,\epsilon^*[$) was treated in [26,27,9] for the specific situation of the Bloch equations.

In this paper, we extend the result in [3] to the new circumstance when $A_k=A+\alpha_k B, \alpha_k\in\mathbb{R}$ and $B_k=B$ (constant) or, equivalently, to the simultaneous controllability of systems submitted to time independent perturbations $dX_k(t)/dt=[A+(u(t)+\alpha_k)B]X_k$. As the result in [3] does not apply to this situation, we prove new controllability results. Moreover, the mathematical techniques

* Corresponding author. Tel.: +216 77 226 575.
E-mail addresses: mohamed.belhadj@ismai.rnu.tn (M. Belhadj), salomon@ceremade.dauphine.fr (J. Salomon), gabriel.turinici@dauphine.fr (G. Turinici).

http://dx.doi.org/10.1016/j.ejcon.2014.12.003
0947-3580/

employed in this work turn out to be useful in other settings such as [49,3] for which we give stronger controllability results.

The perturbation model $A+(u(t)+\alpha_k)B$ was investigated theoretically and numerically in the physical literature independent of any theoretical controllability results. In the quantum computing literature such perturbations are called "fixed systematic errors" (see Section VI.A. Eq. (40) of [22]) or simply "systematic control error", see [23] where the authors concluded that mitigating such errors may be possible (although at the expense of longer pulse sequences). We give here a theoretical result to sustain this view. We also refer to [41], where the authors design pulse sequences that are generically robust with respect to errors in the amplitude of the control field. In a related recent work the corresponding noise model is called "low frequency noise" (see Section IV.C. of [17]): it is defined as the portion of the (control) amplitude noise that has a correlation time that is long (up to 10^3 times) compared to the timescale of the dynamics and as such it can be treated as constant in time. Additional noise models (additive or multiplicative) are presented in [39] in the general quantum control area.

The balance of the paper is as follows: in Section 2 we introduce the general framework and the main notations and in Section 3 we present our main results including a general ensemble controllability result. In Section 4, we apply our results to the controllability of quantum systems. The situation of an infinite number of perturbations is discussed in Section 5 from the theoretical and numerical point of views. Finally, some conclusions and perspectives of future work are given in Section 6.

2. Problem formulation

Let G be a Lie group. Throughout this paper G is considered to be finite dimensional, connected, compact, simple real Lie group. Its Lie algebra is denoted by $\mathfrak{g}$, the identity element is Id and $A, B \in \mathfrak{g}$ are fixed. Remarkable examples of such Lie groups are (see [16,13]) the following:

- the special unitary group SU(N) for $N \geq 2$,
- the special orthogonal group SO(N) for $N \neq 4$,
- the compact symplectic group (quaternionic $N \times N$ unitary matrices) Sp(N) for $N \geq 2$,
- the spin group Spin(N) for $N \geq 2$.

Consider the following control system on G:

$$\frac{dX(t)}{dt} = (A+u(t)B)X(t), \quad X(0) = \mathrm{Id}. \tag{1}$$

The matrix $X(t)$ evolves in the Lie group G.

The controllability of a system on Lie groups such as (1) is a well-studied problem [24,34,12,1,2,48]. The literature on the subject of bilinear control relies essentially on the following theorem (originally due to [20]):

Theorem 1. *Denote by $\mathbb{L}_{A,B}$ the Lie subalgebra of $\mathfrak{g}$ generated by A and B. The system (1) on the Lie group G is controllable if and only if $\mathbb{L}_{A,B} = \mathfrak{g}$ or equivalently if $\dim_{\mathbb{R}} \mathbb{L}_{A,B} = \dim_{\mathbb{R}} \mathfrak{g}$. Moreover there exists $T_{A,B} > 0$ such that any target can be reached in time $t \geq T_{A,B}$ with controls u such that $|u(s)| \leq 1, \forall s \in [0,t]$.*

Here $\dim_{\mathbb{R}} \mathbb{L}_{A,B}$ stands for the dimension of $\mathbb{L}_{A,B}$ as linear vector space over $\mathbb{R}$.

An important question is what happens if the control $u(t)$ in (1) is submitted to some perturbations in a predefined (discrete) list $\{\alpha_k, \ k=1,\dots,K\}$?:

$$\frac{dX_k(t)}{dt} = AX_k(t) + [u(t)+\alpha_k]BX_k(t), \quad X_k(0) = \mathrm{Id}. \tag{2}$$

Can one still control the systems simultaneously? The real perturbation α_k for a given system is not known beforehand, therefore in order to be certain that the system is controlled, one has to find a control $u(t)$ that simultaneously controls all states $X_k(t)$, i.e., find $u(t)$ such that $X_k(T) = V$ for $k=1,\dots,K$ (here V is the target state).

Yet a distinct circumstance is when α_k are not arbitrary perturbations but unknown characteristics of the system to be identified. Here, the goal is to find $u(t)$ such that, given *distinct* V_k one has $X_k(T) = V_k$. By measuring the state of the system at the final time T, one knows which α_k was effective during $[0,T]$.

In conclusion, our problem can be formalized as follows: let $V_k \in G$, $k=1,\dots,K$ be arbitrary. Is it possible to find $T>0$ and a measurable $u:[0,T] \to \mathbb{R}$ such that the system given by (2) satisfies $X_k(T) = V_k, \forall \ k=1,\dots,K$? If the answer to this question is positive then the system in (2) will be called *simultaneously controllable* (or *ensemble controllable*).

3. Simultaneous controllability for perturbations

3.1. Tools for simultaneous controllability

In this section, we recall an important result on simultaneous controllability. Consider K bilinear systems on the (finite dimensional, connected, compact, simple) Lie groups G_k:

$$\frac{dX_k(t)}{dt} = (A_k+u(t)B_k)X_k(t), \quad X_k(0) = \mathrm{Id}, \tag{3}$$

where $A_k, B_k \in \mathfrak{g}_k$, $k=1,\dots,K$ and $\mathfrak{g}_k$ is the Lie algebra of G_k. Recall that when G_k is simple the Lie algebra $\mathfrak{g}_k$ is also simple which means that the only ideals in $\mathfrak{g}_k$ are $\{0\}$ and $\mathfrak{g}_k$. In particular $\mathfrak{g}_k$ is also semi-simple. Let $\mathcal{A} = A_1 \oplus \cdots \oplus A_K \in \oplus_{k=1}^K \mathfrak{g}_k$ and $\mathcal{B} = B \oplus \cdots \oplus B \in \oplus_{k=1}^K \mathfrak{g}_k$. When $\mathfrak{g}_k$ are represented as matrix algebras and $M_k \in \mathfrak{g}_k$ the element $M_1 \oplus \cdots \oplus M_K \in \oplus_{k=1}^K \mathfrak{g}_k$ is simply the block diagonal matrix

$$\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_K \end{pmatrix}.$$

By assembling the K bilinear systems (3), the evolution of this collection of states can be written as a bilinear system on $\oplus_{k=1}^K G_k$:

$$\frac{d\mathbf{X}(t)}{dt} = \mathcal{A}\mathbf{X}(t) + u(t)\mathcal{B}\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{Id} \in \oplus_{k=1}^K G_k. \tag{4}$$

Denote by $\mathbb{L}_{\mathcal{A},\mathcal{B}}$ the Lie algebra generated by the matrices $\mathcal{A}$ and $\mathcal{B}$. Then, we have the following result (see [20,50, Theorems 1 and 2, p. 277] and [28, Section 3] for an application):

Theorem 2. *The collection (3) of K bilinear systems is simultaneously controllable if and only if $\mathbb{L}_{\mathcal{A},\mathcal{B}} = \oplus_{k=1}^K \mathfrak{g}_k$ or equivalently*

$$\dim_{\mathbb{R}} \mathbb{L}_{\mathcal{A},\mathcal{B}} = \sum_{k=1}^K \dim_{\mathbb{R}} \mathfrak{g}_k.$$

Moreover, there exists $T_{\mathcal{A},\mathcal{B}} > 0$ such that any collection of targets $(V_k)_{k=1}^K \in \oplus_{k=1}^K G_k$ can be reached in time $t \geq T_{\mathcal{A},\mathcal{B}}$ with controls $u(t)$ such that $|u(s)| \leq 1, \forall s \in [0,t]$.

3.2. Main result

The proof of our main result uses the following lemma.

Lemma 3. *Consider the collection (3) of K bilinear systems as a control system on $\oplus_{k=1}^K G_k$. Suppose $K>1$ and $\mathbb{L}_{A_k,B_k} = \mathfrak{g}_k$ for any $k=1,\dots,K$. The system is not ensemble controllable if and only if there exist $k,\ell \in \{1,\dots,K\}$, $k \neq \ell$ and an isomorphism $f:\mathfrak{g}_k \to \mathfrak{g}_\ell$ such that $f(A_k) = A_\ell$ and $f(B_k) = B_\ell$.*