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Tutorial on Lyapunov-based methods for time-delay systems ☆

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ABSTRACT

Time-delay naturally appears in many control systems, and it is frequently a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. Therefore, stability and control of time-delay systems is of theoretical and practical importance. Modern control systems usually employ digital technology for controller implementation, i.e. sampled-data control. A time-delay approach to sampled-data control, where the system is modeled as a continuous-time system with the delayed input/output became popular in the networked control systems, where the plant and the controller exchange data via communication network. In the present tutorial, introduction to Lyapunov-based methods for stability of time-delay systems is given together with some advanced results on the topic.

1. Introduction

Time-delay systems (TDSs) are also called systems with after effect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. They belong to the class of *functional differential equations* which are infinite-dimensional, as opposed to ordinary differential equations (ODEs). The simplest example of such a system is

$$\dot{x}(t) = -x(t-h), \quad x(t) \in \mathbb{R},$$

where $h>0$ is the time-delay.

Time-delays appear in many engineering systems – aircraft, chemical control systems, in laser models, in Internet, biology, medicine [31,41]. Delays are strongly involved in challenging areas of communication and information technologies: stability of networked control systems or high-speed communication networks [62].

Time-delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. In the well-known example

$$\ddot{y}(t) + y(t) - y(t-h) = 0,$$

the system is unstable for $h=0$, but is asymptotically stable for $h=1$. The approximation $\dot{y}(t) \simeq [y(t) - y(t-h)]h^{-1}$ explains the damping effect. The stability analysis and robust control of time-delay systems are, therefore, of theoretical and practical importance.

As in systems without delay, an efficient method for stability analysis of TDSs is the Lyapunov method. For TDSs, there exist two main Lyapunov methods: the *Krasovskii* method of Lyapunov *functionals* [43] and the *Razumikhin* method of Lyapunov *functions* [61]. The two Lyapunov methods for linear TDSs result in Linear Matrix Inequalities (LMIs) conditions. The LMI approach to analysis and design of TDSs provides constructive finite-dimensional conditions, in spite of significant model uncertainties [1].

Modern control systems usually employ digital technology for controller implementation, i.e. sampled-data control. Consider a sampled-data control system

$$\dot{x}(t) = Ax(t) + BKx(t_k), \ t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, A, B, K are constant matrices and $\lim_{k\to\infty} t_k = \infty$. This system can be represented as a continuous system with time-varying delay $\tau(t) = t - t_k$ [52,6]:

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)), \quad t \in [t_k, t_{k+1}), \tag{2}$$

where the delay is piecewise-linear (sawtooth) with $\dot{\tau} = 1$ for $t \neq t_k$. Modeling of continuous-time systems with *digital control* in the form of *continuous-time systems with time-varying delay* and the extension of Krasovskii method to TDSs without any constraints on the delay derivative [20] and to discontinuous delays [18] have allowed the development of the *time-delay approach to sampled-data and to network-based control.*

Bernoulli, Euler and Concordet were (among) the first to study equations with delay (the 18-th century). Systematical study started at the 1940s by A. Myshkis and R. Bellman. Since 1960

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there have appeared more than 50 monographs on the subject (see e.g. [5,28,31,41,57] to name a few). The beginning of the 21st century can be characterized as the "time-delay boom" leading to numerous important results. The emphasis in this Introduction to TDSs is on the Lyapunov-based analysis and design of time-delay and sampled-data systems.

The paper is organized as follows. Two main Lyapunov approaches for general TDSs are presented in Section 2. For linear systems with discrete time-varying delays, delay-independent and delay-dependent conditions are provided in Section 3. The section starts from the simple stability conditions and shows the ideas and tools that essentially improve the results. Section 3.3 presents recent Lyapunov-based results for the stability of sampled-data systems. Section 4 considers general (complete) Lyapunov functional for LTI systems with discrete delays corresponding to necessary stability conditions, and discusses the relation between simple, augmented and general Lyapunov functionals. Stability conditions for systems with distributed (finite and infinite) delays are presented in Section 5. Section 6 discusses the stability of some nonlinear systems. Finally the input–output approach to stability of linear TDSs is provided in Section 7 showing the relation of the input–output stability with the exponential stability of the linear TDSs.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi:[-h,0]\to\mathbb{R}^n$, which are absolutely continuous on $[-h,0]$, and have square integrable first order derivatives is denoted by $W[-h,0]$ with the norm $\|\phi\|_W=\max_{\theta\in[-h,0]}|\phi(\theta)|+[\int_{-h}^0|\dot\phi(s)|^2\,ds]^{1/2}$. For $x:\mathbb{R}\to\mathbb{R}^n$ we denote $x_t(\theta)\triangleq x(t+\theta)$, $\theta\in[-h,0]$.

2. General TDS and the direct Lyapunov method

Consider the following TDS:

$$\dot x(t)=f(t,x_t),\quad t\ge t_0,\tag{3}$$

where $f:\mathbb{R}\times C[-h,0]\to\mathbb{R}^n$ is continuous in both arguments and is locally Lipschitz continuous in the second argument. We assume that $f(t,0)=0$, which guarantees that (3) possesses a trivial solution $x(t)\equiv 0$.

Definition 1. The trivial solution of (3) is

- uniformly (in t_0) stable if $\forall t_0\in\mathbb{R}$ and $\forall\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that $\|x_{t_0}\|_C<\delta(\epsilon)$ implies $|x(t)|<\epsilon$ for all $t\ge t_0$;
- *uniformly asymptotically stable* if it is uniformly stable and there exists a $\delta_a>0$ such that for any $\eta>0$ there exists a $T(\delta_a,\eta)$ such that $\|x_{t_0}\|_C<\delta_a$ implies $|x(t)|<\eta$ for all $t\ge t_0+T(\delta_a,\eta)$ and $t_0\in\mathbb{R}$.
- *globally uniformly asymptotically stable* if δ_a can be an arbitrarily large, finite number.

The system is uniformly asymptotically stable if its trivial solution is uniformly asymptotically stable.

Note that the stability notions are not different from their counterparts for systems without delay [36]. In this tutorial we shall only be concerned with uniform asymptotic stability, that sometimes will be referred as asymptotic stability.

Prior to N.N. Krasovskii's papers on Lyapunov functionals and B.S. Razumikhin's papers on Lyapunov functions, L.E. El'sgol'tz (see [5] and references therein) considered the stability problem of the solution $x(t)\equiv 0$ of TDSs by proving that the function $\overline{V}(t)=V(x(t))$ is decreasing in t, where V is some Lyapunov function. This is possible only in some rare special cases. We shall show this on the example of the scalar autonomous Retarded Differential Equation (RDE)

$$\dot x(t)=f(x(t),x(t-h)),\quad f(0,0)=0,$$

where $f(x,y)$ is locally Lipschitz in its arguments. Let us assume that $V(x)=x^2$, which is a typical Lyapunov function for $n=1$. Then we have along the system

$$\frac{d}{dt}[V(x(t))]=2x(t)\dot x(t)=2x(t)f(x(t),x(t-h)).$$

For the feasibility of inequality $(d/dt)[V(x(t))]\le 0$, we need to require that

$$x(t)f(x(t),x(t-h))\le 0$$

for all sufficiently small $|x(t)|$ and $|x(t-h)|$. This essentially restricts the class of equations considered. For example,

$$\dot x(t)=-x(t)x^2(t-h)$$

is stable by the above arguments.

2.1. Lyapunov–Krasovskii approach

Let $V:\mathbb{R}\times C[-h,0]\to\mathbb{R}$ be a continuous functional, and let $x_\tau(t,\phi)$ be the solution of (3) at time $\tau\ge t$ with the initial condition $x_t=\phi$. We define the right upper derivative $\dot V(t,\phi)$ along (3) as follows:

$$\dot V(t,\phi)=\lim\sup_{\Delta t\to 0^+}\frac{1}{\Delta t}[V(t+\Delta t,x_{t+\Delta t}(t,\phi))-V(t,\phi)].$$

Intuitively, a non-positive $\dot V(t,x_t)$ indicates that x_t does not grow with t, meaning that the system under consideration is stable.

Theorem 1 (*Lyapunov–Krasovskii Theorem, Gu et al. [28]*). *Suppose $f:\mathbb{R}\times C[-h,0]\to\mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in $C[-h,0]$) into bounded sets of $\mathbb{R}^n$ and that $u,v,w:\mathbb{R}_+\to\mathbb{R}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s>0$, and $u(0)=v(0)=0$. The trivial solution of (3) is uniformly stable if there exists a continuous functional $V:\mathbb{R}\times C[-h,0]\to\mathbb{R}^+$, which is positive-definite, i.e.*

$$u(|\phi(0)|)\le V(t,\phi)\le v(\|\phi\|_C),\tag{4}$$

and such that its derivative along (3) is non-positive in the sense that

$$\dot V(t,\phi)\le -w(|\phi(0)|).\tag{5}$$

If $w(s)>0$ for $s>0$, then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty}u(s)=\infty$, then it is globally uniformly asymptotically stable.

In some cases functionals $V(t,x_t,\dot x_t)$ that depend on the state-derivatives are useful (see [41, p. 337]). Denote by $W[-h,0]$ the Banach space of absolutely continuous functions $\phi:[-h,0]\to\mathbb{R}^n$ with $\dot\phi\in L_2(-h,0)$ (the space of square integrable functions) with the norm

$$\|\phi\|_W=\max_{s\in[-h,0]}|\phi(s)|+\left[\int_{-h}^0|\dot\phi(s)|^2\,ds\right]^{1/2}.$$

Theorem 1is then extended to continuous functionals

$$V:\mathbb{R}\times W[-h,0]\times L_2(-h,0)\to\mathbb{R}_+,$$

where inequalities (4) and (5) are modified as follows:

$$u(|x(t)|)\le V(t,x_t,\dot x_t)\le v(\|x_t\|_W)\tag{6}$$