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# On some extension of optimal control theory

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ABSTRACT

Some problems of Calculus of Variations do not have solutions in the class of classic continuous and smooth arcs. This suggests the need of a relaxation or extension of the problem ensuring the existence of a solution in some enlarged class of arcs. This work aims at the development of an extension for a more general optimal control problem with nonlinear control dynamics in which the control function takes values in some closed, but not necessarily bounded, set. To achieve this goal, we exploit the approach of R.V. Gamkrelidze based on the generalized controls, but related to discontinuous arcs. This leads to the notion of generalized impulsive control. The proposed extension links various approaches on the issue of extension found in the literature.



## 1. Introduction

It is well known that variational calculus problems may not have a smooth or even continuous solution. Nevertheless, discontinuous solutions may still be of interest from the physical point of view. Consider the following famous Euler example

$$\begin{aligned} &\text{Minimize} \quad \int_0^1 x(t)\sqrt{1+(\dot{x})^2}\,dt, \\ &\text{subject to} \quad x(0)=r_1, \quad x(1)=r_2, \\ &\qquad\qquad\quad x(t)\ge 0. \end{aligned} \tag{1}$$

This is the so-called minimal surface problem. Physically, the solution $x(\cdot)$ is the shape of a soap bubble or a membrane stretched over two parallel disks with radiuses $r_1$ and $r_2$. The application of the Euler–Lagrange principle leads to a second order differential equation and to a boundary-value problem, which does not have solutions for certain values of $r_1, r_2$. The physical meaning is as follows: if numbers $r_1, r_2$ are sufficiently large relatively to the distance between the disks, the membrane exists and the surface of revolution is smooth. But, if we increase the distance between the disks, the soap bubble stretches and, at some point, blows up. At that very moment, the smooth and continuous solution fails to exist. However, it does not mean that a solution $x(\cdot)$ does not exist at all. In this degenerate case, the solution is $x(0)=r_1$, $x(1)=r_2$, $x(t)=0$, $t\in(0,1)$, and, thus, it exhibits discontinuities.

Consider another famous example, the Dido problem.

$$\begin{aligned} &\text{Maximize} \quad \int_0^1 x(t)\,dt, \\ &\text{subject to} \quad x(0)=x(1)=0, \quad x(t)\ge 0, \\ &\int_0^1 \sqrt{1+(\dot{x})^2}\,dt=l. \end{aligned} \tag{2}$$

Once again, continuous solution fails to exist when the length of the arc $l$ is sufficiently large. Dido problem is a typical example of the so-called isoperimetric problem. The situation in which there is no solution is fairly common in such kind of problems.

An isoperimetric version of the Euler example is the catenary

$$\begin{aligned} &\text{Minimize} \quad \int_0^1 x(t)\sqrt{1+(\dot{x})^2}\,dt, \\ &\text{subject to} \quad x(0)=r_1, \quad x(1)=r_2, \quad x(t)\ge 0, \\ &\qquad\qquad\quad \int_0^1 \sqrt{1+(\dot{x})^2}\,dt=l. \end{aligned} \tag{3}$$

The equation of the catenary curve was derived by Leibniz, Huygens and Johann Bernoulli in 1691. They were the first ones to find out that this curve is a hyperbolic cosine, and not a parabola as it had been thought before. The shape of the soap bubble (1) is, as we see from (3), again a catenary, and Euler was the first one to prove it by using Variational Calculus. Once again, this problem naturally allows for discontinuous solutions.

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Another simple example of an isoperimetric problem is the one of minimizing the norm of a function in $\mathbb{L}_1$ over the elements of the unit sphere in $\mathbb{L}_2$

$$\text{Minimize} \quad \int_0^1 |\dot{x}|\, dt,$$
$$\text{subject to} \quad x(0)=0, \quad \int_0^1 |\dot{x}|^2\, dt = 1. \tag{4}$$

The solution does not exist. Indeed the infimum, here, is zero, but it is not reached due to the integral constraint.

In the framework of his famous program, David Hilbert [18] suggested (20th problem) to extend the Variational Calculus in order to cover and to formalize such degenerate situations when solutions fail to exist, by giving a strict mathematical meaning to non-smooth and to non-classical solutions as a whole. He expressed his confidence that every problem in the calculus of variations has a solution, provided that the term *solution* is interpreted appropriately. That spawned a number of theories on the extension of classic variational calculus by various authors. For the rich history of this issue, we refer the reader to the article [21], see also the bibliography therein. Here, we only point out important contributions made by H. Lebesgue, L. Tonelli, L. Young, N. Bogolyubov, R. Gamkrelidze, R. Rockafellar, A. Ioffe, V. Tikhomirov, and J. Warga, among others (see references in the above source for an overview).

So, in the realm of this context, the aim of the current research is to give a strict mathematical meaning to discontinuous solutions that may arise in the following optimal control problem

$$\begin{aligned} &\text{Minimize} && \int_{t_0}^{t_1} g_0(x,v,t)\, dt,\\ &\text{subject to} && \dot{x}=g(x,v,t), \quad t\in[t_0,t_1],\\ &&& x(t_0)\in A, \quad x(t_1)\in B,\\ &&& v(t)\in V \quad \text{a.a. } t\in[t_0,t_1],\\ &&& \varphi(x(t),t)\le 0 \quad \forall t\in[t_0,t_1], \end{aligned} \tag{5}$$

where $g_0:\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^1\to\mathbb{R}^1$, $g:\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^1\to\mathbb{R}^n$, and $\varphi:\mathbb{R}^n\times\mathbb{R}^1\to\mathbb{R}^l$ are given continuous maps, $A$ and $B$ are given closed subsets of $\mathbb{R}^n$, $V$ is a given closed, *not necessarily bounded*, subset of $\mathbb{R}^k$ and $v(\cdot)$ is a control function. The function $\varphi$ defines the so-called state constraints.

In what follows, we associate the control problem (5) an a priori given scalar function $\omega(\xi):\mathbb{R}_+\to\mathbb{R}_+$ which is nonnegative, increasing and continuous. The purpose of introducing this function is to characterize the growth of the dynamics at infinity. Simple examples of $\omega(\xi)$ are: $\xi$ (linear growth), $\xi^2$ (quadratic growth), $\xi^p$, $e^\xi$, etc.

Once $\omega$ is chosen, assume that the admissible control function $v(t)$ in (5) is such that the function $\omega(|v|)$ is integrable. So, when $w(\xi)=\xi$, $v$ is a $\mathbb{L}_1$–function, when $w(\xi)=\xi^2$, $v$ is a $\mathbb{L}_2$–function, etc. Thus, the function $\omega(\xi)$ determines the class of admissible controls in (5).

Note that, by setting $v=\dot{x}$, all the above examples (1)–(4) fall within the formulation of (5), with $V=\mathbb{R}$.

To achieve the above goal, we exploit the extension approach of [13], based on generalized controls, by upgrading it to the case of discontinuous arcs. The case of continuous arcs, however, is still encompassed. Ultimately, the proposed extension links the approach developed by Gamkrelidze with the approaches in [27,28,30], and by some other authors (see [8,20,23]). Moreover, it also generalizes the extensions from [5,6,19], performed for control problems whose dynamics are linear in $v$, with separated control variables.

Overall, the line of investigation undertaken here is, of course, a road well traveled, and based on the so-called "graph completion" or "discontinuous time variable change" technique combined with the Gamkrelidze compactification, or convexification, technique. Besides the above-mentioned sources, our main line of research also goes along the works in [9,10,12,15–17,24,25,29,32]. This list, though, is far from being complete.

## 2. Preliminaries

To treat the extension, it is necessary to compactify the space $\mathbb{R}^k$ by adding a set $S_\infty^{k-1}$, called sphere at infinity. "Sphere at infinity" means that there is a homeomorphism $\Pi: S_\infty^{k-1}\to S^{k-1}$, where $S^{k-1}$ is the unit sphere in $\mathbb{R}^k$. The compactified space $\overline{\mathbb{R}}^k := \mathbb{R}^k\cup S_\infty^{k-1}$ is endowed with a natural topology in which any sequence of points $v_i\in\mathbb{R}^k$ converges to the point $l\in S_\infty^{k-1}$ iff $|v_i|\to\infty$ and

$v_i=|v_i|\cdot\Pi(l)+o(|v_i|)$.

Note that the compact $\overline{\mathbb{R}}^k$ is topologically equivalent to the closed unit ball $B_{\mathbb{R}^k}$ in $\mathbb{R}^k$ due to the following homeomorphism $\Theta$, defined by

$$\Theta(v)=\frac{v}{1+|v|}, \quad v\in\mathbb{R}^k,$$

and $\Theta(v)=\Pi(v)$ if $v\in S_\infty^{k-1}$.

Denote

$$V_\infty := \Pi^{-1}\left(\operatorname*{Limsup}_{|v|\overset{V}{\to}\infty}\frac{v}{|v|}\right).$$

Here $|v|\overset{V}{\to}\infty$ means that $|v|\to\infty$ and $v\in V$.

Now, the set $\overline{V}:=V\cup V_\infty$ is compact.

Let us introduce our main hypothesis about the right-hand side $g$ and function $g_0$. Assume that there exists a continuous function

$g^\infty:\mathbb{R}^n\times S^{k-1}\times\mathbb{R}^1\to\mathbb{R}^n$,

such that

$$\lim_{v\to\Pi^{-1}(e)}\frac{g(x,v,t)}{\omega(|v|)}=g^\infty(x,e,t) \quad \forall\ x,t\in\mathbb{R}^n\times\mathbb{R}^1, \ \ \forall\ e\in S^{k-1}.$$

Then, there is defined a continuous function $\overline{g}:\mathbb{R}^n\times\overline{\mathbb{R}}^k\times\mathbb{R}^1\to\mathbb{R}^n$ such that

$$\overline{g}(x,v,t)=\frac{g(x,v,t)}{1+\omega(|v|)} \quad \text{if } v\in\mathbb{R}^k,$$

and $\overline{g}(x,v,t)=g^\infty(x,\Pi(v),t)$, if $v\in S_\infty^{k-1}$.

In a similar way, assume that there exist a function $g_0^\infty:\mathbb{R}^n\times S^{k-1}\times\mathbb{R}^1\to\mathbb{R}^1$, and a function $\overline{g}_0:\mathbb{R}^n\times\overline{\mathbb{R}}^k\times\mathbb{R}^1\to\mathbb{R}^1$ which are defined just as above but using $g_0$ instead of $g$.

We shall use the following assumptions.

(H1) Functions $\overline{g}$, and $\overline{g}_0$ introduced above do exist. Moreover, the function $\overline{g}$ is such that

$$|\overline{g}(x,v,t)|\le m(t)\kappa(|x|) \quad \forall(x,v,t)\in\mathbb{R}^n\times\overline{V}\times\mathbb{R}^1,$$

where $\kappa:\mathbb{R}^1\to\mathbb{R}^1$ is such that

$$\frac{\kappa(|x|)}{1+|x|}\le \text{const} \quad \forall x,$$

and $m$ is some locally integrable function.

(H2) Functions $\overline{g}$, and $\overline{g}_0$ are continuously differentiable in $x,t$ for all $v\in\overline{\mathbb{R}}^k$.

**Definition 1.** The control problem (5) is said to allow the impulsive extension of order $\omega$ provided the hypothesis (H1) is satisfied and at least one of the functions $g^\infty$ or $g_0^\infty$ is not a zero function.

It is easy to see that the Euler and Dido problems (1), (2), and (3) allow extension of linear order $\xi$ whereas the problem (4) already requires quadratic growth $\omega(\xi)=\xi^2$.

Consider a scalar Borel measure $\mu:\mathcal{B}(T)\to[0,+\infty)$, $T=[t_0,t_1]$. Here, $\mathcal{B}(T)$ stands for the $\sigma$-algebra of Borel subsets of $T$.