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Repeated eigenstructure assignment for controlled invariant subspaces[☆]

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ABSTRACT

This paper is concerned with the computation of basis matrices for the subspaces that lie at the core of the so-called geometric approach to control theory, namely the supremal output-nulling, reachability and stabilisability subspaces. Importantly, we also consider the problem of computing the feedback matrices that render these subspaces invariant with respect to the closed loop, while simultaneously assigning the assignable eigenstructure of the closed loop. Differently from the classical techniques presented in the literature so far on this topic, which are based on the standard pole assignment algorithms and are therefore applicable only in the non-defective case, the method presented in this paper can be applied in the case of closed-loop eigenvalues with arbitrary multiplicity.

1. Introduction

In the last 40 years, geometric control has played a fundamental role in the understanding of the structural properties of dynamical systems and in the solution of several control and estimation problems, including disturbance decoupling, non-interacting control, fault detection, model matching and optimal control to name a few. The monographs [19,4,18,6] provide surveys of the extensive literature in this area.

The subspaces that underpin the classic geometric theory of linear time-invariant (LTI) systems are the so-called controlled invariant and output-nulling subspaces (and their duals). The most important output-nulling subspace is undoubtedly $\mathcal{V}^\star$, which represents the set of initial states for which a control function exists that maintains the output function identically at zero; the second is $\mathcal{R}^\star$, which represents the reachable subspace within $\mathcal{V}^\star$ when the output is constrained to be identically zero. Finally, the subspace $\mathcal{V}_g^\star$ represents the set of initial states for which a control can be found that maintains the output at zero by means of state trajectories that have dynamics belonging to a "good" region of the complex plane. In the LTI case, the input functions that maintain the state trajectory on output-nulling subspaces and the output at zero can always be expressed as a static state feedback, by means of a feedback matrix usually referred to as a *friend* of the output-nulling subspace.[1]

The computation of friends of output-nulling subspaces that assign the inner and outer assignable spectrum of the closed-loop has been considered by many authors and the texts [4,6] included publicly available MATLAB® toolboxes. In the MATLAB® *GA* toolbox,[2] the `effesta.m` routine is used for computing the friends. Similarly, the *special coordinate basis* method of [6] was incorporated into the computation of the friends in the MATLAB® *Linsyskit* toolbox[3]; the `atea.m` routine is used for computing the friends, and is described in [9].

All the methods currently available in the literature are based on decompositions that reduce the problem to one where a feedback matrix F is sought that assigns all the eigenvalues of a closed-loop matrix, say $A+BF$, where the pair (A,B) is completely reachable. Both the methods in the MATLAB® toolboxes *GA* and *Linsyskit* exploit the MATLAB® instruction `place.m` to this purpose, based on the algorithm of [7], which can only assign eigenvalues of $A+BF$ with a multiplicity that must not exceed the rank of B (which correspond to a trivial – i.e., diagonal – Jordan

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[1] This property does not necessarily hold outside the framework of finite dimensional LTI systems over a field.
[2] The MATLAB® geometric approach (*GA*) toolbox is available at http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm.
[3] The Linear System Toolkit is available on request from the first author of [6]; see http://vlab.ee.nus.edu.sg/~bmchen.

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form for the closed loop). This limitation of the routine `place.m` is thus inherited by the MATLAB® instructions of the toolboxes *GA* and *Linsyskit*, which can therefore compute the friend of output-nulling subspaces $\mathcal{V}^\star$, $\mathcal{R}^\star$ and $\mathcal{V}_g^\star$ only in the case of non-defective closed-loop.

The same limitation is found in the method in [11], where an algorithm is presented for the computation of bases for $\mathcal{R}^\star$ and $\mathcal{V}^\star$ based on the computation of the null-spaces of the system Rosenbrock matrix pencil [14]. This procedure has the advantage of simultaneously delivering, as a by-product, a corresponding friend F that assigns a certain inner closed-loop eigenstructure. In [12,13] the method was extended to include the computation of a friend that also assigns the free outer eigenstructure of $\mathcal{R}^\star$ (or $\mathcal{V}^\star$).

One of the restrictive assumptions of the method proposed in [11], which remains in the generalisations presented in [12,13], is the fact that the closed-loop eigenvalues to be assigned must be distinct. The first contribution of the current paper is to extend the method in [11] to accommodate the problem of assigning repeated closed-loop eigenvalues, with arbitrary multiplicity. This task is accomplished by introducing a new parameterisation of the basis matrices for $\mathcal{R}^\star$, $\mathcal{V}^\star$ and $\mathcal{V}_g^\star$, and this provides a natural method for determining the associated friend which can place the assignable closed-loop eigenvalues to desired locations virtually without any assumptions on the location or on the multiplicity of such eigenvalues.

However, the most important aspect of the method presented in [11], which has remained unexploited until very recent times, is the fact that the friend of $\mathcal{R}^\star$ (or $\mathcal{V}^\star$, or $\mathcal{V}_g^\star$) that assigns the free inner and outer eigenstructure of the closed-loop with respect to $\mathcal{R}^\star$ is given in parameterised form. This fundamental aspect invites the formulation of optimisation problems aimed at exploiting the available freedom to deal with performance objectives. The *robust exact pole placement problem* [15] involves obtaining a state feedback matrix F that assigns a desired set of closed-loop eigenvalues while also rendering them as insensitive to perturbations in A, B, C, D and F as possible. The *minimum gain exact pole placement problem* [15] involves assigning a desired set of closed-loop eigenvalues with the friend that has the least matrix norm (gain). For systems without outputs, there has been considerable literature on these problems. Papers considering the robust exact pole placement problem for the case of a possibly defective eigenstructure include [8,1,16]. For the minimum gain exact pole placement problem, [3,16] considered the general problem of assigning any desired set of poles with any desired multiplicities with minimum Frobenius gain.

However, as these papers considered systems without output components, the methods are not applicable to the optimal computation of friends of controlled invariant subspaces. The papers [12,13] were the first to propose a method for computing friends that yield a robust eigenstructure, or have the least gain. Thus, the second contribution of this paper is the extension of the methods of [12,13] to accommodate the optimal computation of friends that assign any desired eigenstructure.

Notation. Throughout this paper, the symbol $\{0\}$ stands for the origin of a vector space. For convenience, a linear mapping between finite-dimensional spaces and a matrix representation with respect to a particular basis are not distinguished notationally. The image and the kernel of matrix A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. The Moore–Penrose pseudo-inverse of A is denoted by $A^\dagger$. When A is square, we denote by $\sigma(A)$ the spectrum (i.e., the Jordan structure) of A. Given a linear map $A:\mathcal{X}\longrightarrow\mathcal{Y}$ and a subspace $\mathcal{S}$ of $\mathcal{Y}$, the symbol $A^{-1}\mathcal{S}$ stands for the inverse image of $\mathcal{S}$ with respect to the linear map A, i.e., $A^{-1}\mathcal{S}=\{x\in\mathcal{X}\,|\,Ax\in\mathcal{S}\}$. If $\mathcal{J}\subseteq\mathcal{X}$, the restriction of the map A to $\mathcal{J}$ is denoted by $A|\mathcal{J}$. If $\mathcal{X}=\mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenstructure of the map A restricted to $\mathcal{J}$ is denoted by $\sigma(A|\mathcal{J})$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are A-invariant subspaces and $\mathcal{J}_1\subseteq\mathcal{J}_2$, the mapping induced by A on the quotient space $\mathcal{J}_2/\mathcal{J}_1$ is denoted by $A|\mathcal{J}_2/\mathcal{J}_1$, and its spectrum is denoted by $\sigma(A|\mathcal{J}_2/\mathcal{J}_1)$. Given a map $A:\mathcal{X}\longrightarrow\mathcal{X}$ and a subspace $\mathcal{Y}$ of $\mathcal{X}$, we denote by $\langle A,\mathcal{Y}\rangle$ the smallest A-invariant subspace of $\mathcal{X}$ containing $\mathcal{Y}$. The symbol i stands for the imaginary unit, i.e., $\mathrm{i}=\sqrt{-1}$. The symbol $\overline{\alpha}$ denotes the complex conjugate of $\alpha\in\mathbb{C}$.

2. Preliminaries

In what follows, whether the underlying system evolves in continuous or discrete time is irrelevant and, accordingly, the time index set of any signal is denoted with the symbol $\mathbb{T}$, which represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane $\mathbb{C}^-$ in the continuous time or the open unit disc $\mathbb{C}^\circ$ in the discrete time. Consider a linear time-invariant system Σ governed by

$$\Sigma:\begin{cases}\mathcal{D}x(t)=Ax(t)+Bu(t), & x(0)=x_0,\\ y(t)=Cx(t)+Du(t),\end{cases}\tag{1}$$

where, for all $t\in\mathbb{T}$, $x(t)\in\mathcal{X}=\mathbb{R}^n$ is the state, $u(t)\in\mathcal{U}=\mathbb{R}^m$ is the control input, $y(t)\in\mathcal{Y}=\mathbb{R}^p$ is the output, and A, B, C and D are appropriate dimensional constant real-valued matrices. The operator $\mathcal{D}$ denotes either the time derivative in the continuous time, i.e., $\mathcal{D}x(t)=\dot{x}(t)$, or the unit time shift in the discrete time, i.e., $\mathcal{D}x(t)=x(t+1)$. Let the system Σ described by (1) be identified with the quadruple (A,B,C,D). We assume with no loss of generality that all the columns of $\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]$ and all the rows of $[C\ D]$ are linearly independent.[4] We define the Rosenbrock matrix as the matrix pencil:

$$P_\Sigma(\lambda)\overset{\text{def}}{=}\begin{bmatrix}A-\lambda I_n & B\\ C & D\end{bmatrix},\tag{2}$$

in the indeterminate $\lambda\in\mathbb{C}$ [14]. The invariant zeros of Σ are identified with the values of $\lambda\in\mathbb{C}$ for which the rank of $P_\Sigma(\lambda)$ is strictly smaller than its normal rank.[5] More precisely, the invariant zeros are the roots of the non-zero polynomials on the principal diagonal of the Smith form of $P_\Sigma(\lambda)$, see [2]. Given an invariant zero $\lambda=z\in\mathbb{C}$, the rank deficiency of $P_\Sigma(\lambda)$ at the value $\lambda=z$ is the geometric multiplicity of the invariant zero z, and is equal to the number of elementary divisors (invariant polynomials) of $P_\Sigma(\lambda)$ associated with the complex frequency $\lambda=z$. The degree of the product of the elementary divisors of $P_\Sigma(\lambda)$ corresponding to the invariant zero z is the algebraic multiplicity of z, see [10]. More explicitly, denoting the invariant zeros of (2) as $z_1,\ldots,z_\eta$ and denoting the elementary divisors of $P_\Sigma(\lambda)$ by

$$\gamma_k(\lambda)=(\lambda-z_1)^{m_{1,k}}(\lambda-z_2)^{m_{2,k}}\cdots(\lambda-z_\eta)^{m_{\eta,k}},\quad k\in\{1,\ldots,c\},\tag{3}$$

(where c is the number of elementary divisors) ordered in such a way that $m_{k,c}\ge m_{k,c-1}\ge\cdots\ge m_{k,2}\ge m_{k,1}$ for any $k\in\{1,\ldots,\eta\}$, the geometric multiplicity of the invariant zero z_i equals the number of $m_{i,j}\neq 0$ for $j\in\{1,\ldots,c\}$, while the algebraic multiplicity of z_i is equal to $\sum_{k=1}^{c} m_{i,k}$. Thus, the algebraic multiplicity of an invariant zero is not smaller than its geometric multiplicity.

Before introducing the geometric tools that will be needed in this paper, we recall that if the pair (A,B) is completely reachable (i.e., the subspace $\mathcal{R}_0\overset{\text{def}}{=}\operatorname{im}[B\ AB\ \cdots\ A^{n-1}B]=\langle A,\operatorname{im}B\rangle$ coincides with $\mathcal{X}$), a feedback matrix $F\in\mathbb{R}^{m\times n}$ exists such that all the eigenvalues of the closed-loop matrix $A+BF$ are freely assignable

[4] If $\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]$ has non-trivial kernel, a subspace $\mathcal{U}_0$ of the input space exists that does not influence the state dynamics. By performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent system for which this condition is satisfied. Likewise, if $[C\ D]$ is not surjective, there are some outputs that result as linear combinations of the remaining ones, and these can be eliminated using a dual argument by performing a change of coordinates in the output space.

[5] The normal rank of a rational matrix $M(\lambda)$ is defined as $\operatorname{normrank} M(\lambda)\overset{\text{def}}{=}\max_{\lambda\in\mathbb{C}}\operatorname{rank}M(\lambda)$. The rank of $M(\lambda)$ is equal to its normal rank for all but finitely many $\lambda\in\mathbb{C}$.