European Journal of Control 26 (2015) 22–27

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

Periodic open-loop stabilization of planar switched systems

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ARTICLE INFO

Article history:
Received 3 November 2014
Received in revised form
2 September 2015
Accepted 3 September 2015
Recommended by A. Astolfi
Available online 18 September 2015

Keywords:
Stabilization
Switched systems
Periodic switching signals

ABSTRACT

In the context of the theory of switched systems, and especially of the open-loop stabilization problem, it is interesting to study the relationship between the placement of the eigenvalues of a matrix of the form $H=\theta_1A_1+\theta_2A_2$ and those of the matrix $E=e^{\theta_2A_2}e^{\theta_1A_1}$. It is well known that if all the eigenvalues of H have negative real part and $\theta_1+\theta_2$ is small enough, then the eigenvalues of E lie in the unit disc of the complex plane. In this paper we prove that in the two dimensional case a partial converse holds: if the eigenvalues of E lie in the unit disc of the complex plane for sufficiently small values of $\theta_1+\theta_2$, then there exist some τ_1,τ_2 (with, in general, $\tau_1 \neq =\theta_1, \tau_2 \neq =\theta_2$) such that the eigenvalues of the matrix $\tau_1A_1+\tau_2A_2$ have negative real part.

1. Introduction and notation

The problem addressed in this paper deals with the existence of fast switching, uniform, periodic open-loop stabilizers i.e., switching signals periodic of arbitrarily small period, which render asymptotically stable a given switched system. This is a particular instance of the asymptotic controllability problem (also called consistent stabilization) recently studied in [3,10,11]. Here, we limit ourselves to switched systems composed by pairs of linear vector fields $f_1(x)=A_1x$, $f_2(x)=A_2x$, where A_1,A_2 are real $d\times d$ matrices and, for the moment, d is any integer, $d\geq 1$. A well know approach to this problem is based on the examination of the eigenvalues of the matrix $H(\alpha)=\alpha A_1+(1-\alpha)A_2$, $\alpha\in[0,1]$. In particular, it is well known that a solution of the problem exists whenever the following condition is met:

($\mathcal{H}$) There exists a number $\alpha\in[0,1]$ such that all the eigenvalues of the matrix $H(\alpha)$ have negative real part.

More precisely, a periodic switching signal can be constructed, under Assumption ($\mathcal{H}$), according to the following rule:

($\mathcal{R}$) Let the periodicity interval $[0,T]$ be divided into two parts, respectively, proportional to α and $1-\alpha$, and let the active vector field be f_1 on the first part, and f_2 on the second part of the interval.

Such a switching signal uniformly "stabilizes" the system in the sense that the corresponding switched trajectories asymptotically approach the origin for each initial state, provided that T is small enough. In fact, under these conditions the convergence of the switched trajectories is exponential [11].

It is natural to ask the question whether Condition ($\mathcal{H}$) is necessary, as far as sufficient, for the existence of periodic switching signals of arbitrarily small period, which stabilize the system in the aforementioned sense. In this paper we prove that when $d=2$, the answer is basically positive but the values of α for which Condition ($\mathcal{H}$) is fulfilled do not coincide, in general, with the values of α for which a periodic stabilizing switching signal can be constructed according to the rule ($\mathcal{R}$).

Moreover, we show that the construction of periodic open-loop switching stabilizers of arbitrarily small period can be achieved also in some critical cases, where a number α can be found in such a way that all the eigenvalues of $H(\alpha)$ have non-positive real parts, and the real part of at least one eigenvalue of $H(\alpha)$ is zero.

A more formal exposition of the problem and of the main result will be given in Section 2. Section 3 contains the proof, and some examples are discussed in Section 4. We end this introduction with some comments about the role and history of assumption ($\mathcal{H}$) in the literature.

Condition ($\mathcal{H}$) appears for the first time in a paper by Tokarzewski [12], where it is used to construct fast switching, periodic, open-loop stabilizers for a finite family of linear vector fields $\{f_n(x)=A_nx, n=1,\dots,N\}$, according to the rule ($\mathcal{R}$). Later, Condition ($\mathcal{H}$) was independently re-proposed by Wicks et al. [14], where they prove that it implies quadratic stabilization, namely the

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http://dx.doi.org/10.1016/j.ejcon.2015.09.002
0947-3580/

existence of a definite positive, symmetric $d \times d$ matrix P such that

$$\forall x \neq = 0 \ \exists n \in \{1, \ldots, N\} \quad \text{such that } x^{\mathbf{t}}(A_n^{\mathbf{t}} P + P A_n) x < 0. \tag{1}$$

Quadratic stabilization leads in turn to the construction of a closed-loop stabilizer, represented by a discontinuous state-dependent switching rule. In [14], the authors need to introduce hysteresis in order to counteract possible chattering phenomena along the discontinuity set of the switching rule. Alternatively, the stability could be intended in the sense of Filippov solutions (see [1] and the so-called min-projection strategy proposed in [9]). This is basically equivalent to interpret the closed-loop system resulting from the application of the discontinuous switching rule as a differential inclusion. However, resorting to Filippov solutions has a drawback: indeed in general Filippov solutions cannot be reproduced by means of time dependent switched signals.

In general, Condition $(\mathcal{H})$ is not necessary for the existence of discontinuous stabilizing state dependent switching rules (see the discussion in [1], where it is also given a generalized form of Condition $(\mathcal{H})$). On the other hand, it has been proven that if $\mathcal{F}$ is formed by a pair of matrices (i.e., $N = 2$), Condition $(\mathcal{H})$ is actually necessary for quadratic stabilization [8]. However, in general this last result does not hold when $N > 2$: indeed, in [16] the authors exhibit an example of a family formed by three 2×2-matrices which is quadratically stabilizable but does not satisfy assumption $(\mathcal{H})$. Tokarzewski's result (reported below as Theorem 1) has been recently brought to the attention of the control theorists community by the book [11]. Tokarzewski obtained also a partial converse of Theorem 1 (reported below as Proposition 1). A nonlinear extension of Tokarzewski's result has been obtained in [3].

Throughout this paper, we adopt the following notation: the pair of matrices A_1, A_2 is denoted by $\mathcal{F}$. A switching signal (i.e., any piecewise, right continuous function from $[0, +\infty)$ to $\{1, 2\}$) is denoted by σ. The switched system defined by $\mathcal{F}$ and σ is denoted by $(\mathcal{F}, \sigma)$. For any pair of nonnegative numbers θ_1, θ_2, we denote $\Phi(\theta_1, \theta_2) = e^{\theta_2 A_2} e^{\theta_1 A_1}$. We will use also the following shortened notation: $E(T, \alpha) = \Phi(T\alpha, T(1-\alpha))$, $\tilde{A}_1 = \alpha A_1$, $\tilde{A}_2 = (1-\alpha) A_2$.

Finally, recall that a matrix M is said to be *Hurwitz* when all its eigenvalues have negative real part; *Schur* when all its eigenvalues lie in the unit open disc. In addition, we agree to call *anti-Hurwitz* a matrix M such that $-M$ is Hurwitz.

2. Description of the problem and main result

It is well known that the asymptotic behavior of a switched system formed by linear vector fields and driven by a periodic switching signal (the same for each initial state) is basically equivalent to the asymptotic behavior of an associated discrete time dynamical system [4]. More precisely, for a given pair of matrices $\mathcal{F} = \{A_1, A_2\}$, let us take a pair of positive numbers θ_1, θ_2, and let σ be the switched signal, periodic of period $T = \theta_1 + \theta_2$, such that

$$\sigma(t) = \begin{cases} 1 & \text{for } t \in [0, \theta_1) \\ 2 & \text{for } t \in [\theta_1, \theta_1 + \theta_2). \end{cases} \tag{2}$$

Then, the following statements are equivalent:

(i) The switched system $(\mathcal{F}, \sigma)$ is asymptotically stable at the origin.

(ii) The time varying (periodic, piecewise constant) linear system $\dot{x} = A(t)x$, where $A(t) = A_{\sigma(t)}$, is asymptotically stable at the origin.

(iii) The discrete dynamical system

$$\xi_{k+1} = \Phi(\theta_1, \theta_2)\xi_k. \tag{3}$$

is asymptotically stable at the origin.

(iv) The matrix $\Phi(\theta_1, \theta_2)$ is Schur.

According to the previous equivalences, the periodic stabilization problem for $\mathcal{F}$ reduces to an algebraic problem. The following sufficient condition is well known [15,12,11].

Theorem 1. *Let $\mathcal{F}$ be a pair of matrices. Let $\alpha \in [0, 1]$ be such that the matrix $H(\alpha) = \alpha A_1 + (1-\alpha) A_2 = \tilde{A}_1 + \tilde{A}_2$ is Hurwitz. Then, there exists $T_0 > 0$ such that the matrix*

$$E(T, \alpha) = e^{T(1-\alpha)A_2} e^{T\alpha A_1} = e^{T\tilde{A}_2} e^{T\tilde{A}_1} \tag{4}$$

(with the same α) is Schur for each positive $T \leq T_0$.

The proof of Theorem 1 is straightforward if A_1, A_2 commute (this is for instance the case of the following Example 1). Indeed, in this case, one has

$$e^{T\tilde{A}_2} e^{T\tilde{A}_1} = e^{T(\tilde{A}_1 + \tilde{A}_2)}$$

and the left hand side is recognized to be Schur for each $T > 0$, since the eigenvalues of an exponential matrix e^M are just the exponential of the eigenvalues of M.

Example 1. Consider the family $\{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}.$$

It is straightforward to verify that the convex combination $(A_1 + A_2)/2$ is Hurwitz and that

$$E(T, 1/2) = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{-T} \end{pmatrix}$$

is Schur for each $T > 0$.

In the much more frequent case where A_1, A_2 do not commute, one way to prove Theorem 1 is to use the Baker–Campbell–Hausdorff formula (see for instance [13]), which allows us to write $e^{T\tilde{A}_2} e^{T\tilde{A}_1} = e^{C(T)}$ where

$$C(T) = T(\tilde{A}_1 + \tilde{A}_2) + \frac{T^2}{2}[\tilde{A}_2, \tilde{A}_1] + \frac{T^3}{12}([\tilde{A}_2, [\tilde{A}_2, \tilde{A}_1]] - [\tilde{A}_1, [\tilde{A}_2, \tilde{A}_1]]) + \cdots \tag{5}$$

and $[M, N] = MN - NM$ denotes the commutator of the matrices M, N. If $\tilde{A}_1 + \tilde{A}_2$ is Hurwitz, then for small T the terms of order greater than one can be neglected, and the reasoning can be easily carried out. However in general, as the following example shows, the matrix (4) might be not Schur for large T.

Example 2. Consider the family $\{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

The convex combination $(A_1 + A_2)/2$ is Hurwitz. It is not difficult to compute

$$E(T, 1/2) = \begin{pmatrix} e^{T/2} \cos T & -e^{T/2} \sin T \\ e^{-T} \sin T & e^{-T} \cos T \end{pmatrix}$$

whose characteristic equation is

$$\lambda^2 - \lambda(e^{-T} + e^{T/2}) \cos T + e^{-T/2} = 0.$$

The roots can be computed and analyzed possibly with the aid of a symbolic algebra package. Then their graphs can be plotted. The conclusions can be resumed in this way. There exist two numbers $0 < T_1 < T_2$ such that:

- for $0 < T < T_1$, $E(T, 1/2)$ is Schur, with conjugate complex eigenvalues;