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Scalable control of positive systems

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ABSTRACT

Classical control theory does not scale well for large systems such as power networks, traffic networks and chemical reaction networks. However, many such applications in science and engineering can be efficiently modeled using the concept of positive systems and the nonlinear counterpart monotone systems. It is therefore of great interest to see how such models can be used for control.

This paper demonstrates how positive systems can be exploited for analysis and design of large-scale control systems. Methods for synthesis of distributed controllers are developed based on linear Lyapunov functions and storage functions instead of quadratic ones. The main results are extended to frequency domain input–output models using the notion of positively dominated system. Applications to transportation networks and vehicle formations are provided.

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1. Introduction

The theory of positive systems and nonnegative matrices has a long history, dating back to the Perron–Frobenius Theorem in 1912. A classic book on the topic is [2]. The theory is used in Leontief economics [11], where the states denote nonnegative quantities of commodities. It also appears in the study of Markov chains [21], where the states denote nonnegative probabilities and in compartment models [9] to model for example populations of species. A nonlinear counterpart is the theory for monotone systems, characterized by the property that a partial ordering of initial states is preserved by the dynamics. Such dynamical systems were studied in a series of papers by Hirsch [13,14].

Positive systems have also been studied in the control literature [23,6,10], and increasingly so during the last decade. Feedback stabilization of positive linear systems was studied in [5]. Stabilizing static output feedback controllers were parameterized using linear programming in [16,15] and extensions to input–output gain were given in [24,4]. Tanaka and Langbort [22] proved that the input–output gain of positive systems can be evaluated using a diagonal quadratic storage function and utilized this for H_∞ optimization of decentralized controllers in terms of semi-definite programming.

This paper builds on several contributions by the author [17–19], deriving theory that is applicable to control systems of very large scale. Such systems appear for example in traffic networks, power networks and chemical reaction networks. Classical methods for multi-variable control, such as linear quadratic control and H_∞-optimization, do not scale well. The difficulties are partly due to computational complexity and partly to the absence of distributed structure in the resulting controllers. The complexity growth can be traced back to the fact that stability verification of a linear system with n states generally requires a Lyapunov function involving n^2 quadratic terms, even if the system matrices are sparse. As was pointed out in [17], the situation improves drastically if we restrict attention to positive systems. Then stability and input–output gain can be verified using a Lyapunov function with only n linear terms. Sparsity can be exploited and even synthesis of distributed controllers can be done with a complexity that grows linearly with the number of nonzero entries in the system matrices. As will be demonstrated in this paper, these observations have far-reaching implications for control engineering:

1. The conditions that enable scalable solutions hold naturally in many important application areas, such as stochastic systems, economics, transportation networks, chemical reactions, power systems and ecology.
2. The essential mathematical property can be extended to frequency domain models, using the concept of “positively dominated” transfer function.
3. The assumption of positive dominance need not hold for the open loop process. Instead, a large-scale control system can often be structured into local control loops that give positive dominance, thus enabling scalable methods for optimization of global input–output gains.

The paper is structured as follows: Section 2 introduces notation. Stability conditions and input–output bounds for positive systems are reviewed in Section 3. Those results are mostly well known, but some aspects of Proposition 3–5 are new. The use in

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scalable verification of transportation networks and vehicle formations is introduced in Section 4. Similar ideas are then exploited in Section 5 for synthesis of stabilizing optimal controllers using distributed linear programming. The contributions, Theorems 6 and 7, can be viewed as generalizations of results in [4]. Finally, Section 6 extends the techniques to frequency domain input–output models using the notion of positively dominated transfer function. Further applications to vehicle platoons and transportation networks are given. Summarizing conclusions are given before an appendix with proofs and references.

2. Notation and terminology

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. For $x \in \mathbb{R}^n$, let $|x| \in \mathbb{R}^n_+$ be the element-wise absolute value. This notation should not be confused with the vector norms $|w|_p = (|w_1|^p + \cdots + |w_m|^p)^{1/p}$ for $p \in (0, \infty)$ and $|w|_\infty = \max_i |w_i|$. Given $M \in \mathbb{C}^{r \times m}$, define the induced matrix norm

$$\|M\|_{p-\text{ind}} = \sup_{w \in \mathbb{R}^m \setminus \{0\}} \frac{|Mw|_p}{|w|_p}.$$

The spectral norm $\|M\|_{2-\text{ind}}$ will often just be denoted $\|M\|$. Let $\theta(t)$ be the Heaviside step function and $\delta(t)$ the Dirac delta function. Then the transfer matrix $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ has the impulse response $g(t) = Ce^{At}B\theta(t) + D\delta(t)$. With $w \in \mathbf{L}^m_p[0, \infty)$, let $g * w \in \mathbf{L}^r_p[0, \infty)$ be the convolution of g and w and define the induced norms

$$\|g\|_{p-\text{ind}} = \sup_{w \in \mathbf{L}^m_p[0,\infty)} \frac{\|g * w\|_p}{\|w\|_p}$$

where $\|w\|_p = \left(\sum_k \int_0^\infty |w_k(t)|^p \, dt\right)^{1/p}$ for $p \in (0, \infty)$ and $\|w\|_\infty = \sup_t \max_k |w_k(t)|$. Let $\|\mathbf{G}\|_{p-\text{ind}} := \|g\|_{p-\text{ind}}$, where g is the impulse response of $\mathbf{G}$. The norm $\|\mathbf{G}\|_{2-\text{ind}}$ is often called the H_∞ norm, denoted $\|\mathbf{G}\|_\infty$. It is well known that $\|\mathbf{G}\|_\infty = \sup_\omega \|\mathbf{G}(i\omega)\|$.

The notation $\mathbf{1}$ means a column vector with all entries equal to one. The inequality $X > 0$ ($X \geq 0$) means that all elements of the matrix (or vector) X are positive (nonnegative). For a symmetric matrix X, the inequality $X \succ 0$ means that the matrix is positive definite. The matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if all eigenvalues have negative real part. It is *Schur* if all eigenvalues are strictly inside the unit circle. Finally, the matrix is said to be *Metzler* if all off-diagonal elements are nonnegative. The notation $\mathbb{RH}_\infty$ represents the set of rational functions with real coefficients and without poles in the closed right half plane. The set of $n \times m$ matrices with elements in $\mathbb{RH}_\infty$ is denoted $\mathbb{RH}^{n \times m}_\infty$. The state space model

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

is said to be an *internally positive system* if A is Metzler and $B, C, D \geq 0$. It is called an *externally positive system* if the impulse response $Ce^{At}B\theta(t) + D\delta(t)$ is nonnegative. A transfer matrix $\mathbf{G}$ is called *positively dominated* if every matrix entry satisfies $|\mathbf{G}_{jk}(i\omega)| \leq \mathbf{G}_{jk}(0)$ for all $\omega \in \mathbb{R}$.

The term positive system will not be given a precise definition. However, internally positive systems, externally positive systems and positively dominated systems will all be viewed as instances of positive systems. The essential property is that there exists a positive cone in the signal space which is left invariant by the input–output map.

3. Preliminaries

This section introduces some preliminary results on positive systems. Propositions 1 and 2 are well known in the literature since before. References are given in the appendix. Propositions 3–5 are partly known from [4], partly new. Full proofs are given in the same appendix.

Proposition 1. *Given a Metzler matrix $A \in \mathbb{R}^{n \times n}$, the following statements are equivalent*:

(1.1) *The matrix A is Hurwitz.*
(1.2) *There exists $\xi \in \mathbb{R}^n$ such that $\xi > 0$ and $A\xi < 0$.*
(1.3) *There exists $z \in \mathbb{R}^n$ such that $z > 0$ and $z^T A < 0$.*
(1.4) *There exists a diagonal matrix $P \succ 0$ such that $A^T P + PA \prec 0$.*
(1.5) *The matrix $-A^{-1}$ exists and has nonnegative entries.*

Moreover, if $\xi = (\xi_1, \ldots, \xi_n)$ and $z = (z_1, \ldots, z_n)$ satisfy the conditions of (1.2) *and* (1.3) *respectively, then $P = \text{diag}(z_1/\xi_1, \ldots, z_n/\xi_n)$ satisfies the conditions of* (1.4).

Remark 1. Each of the conditions (1.2), (1.3) and (1.4) corresponds to a Lyapunov function of a specific form. If $A\xi < 0$, then $V(x) = \max_i(|x_i|/\xi_i)$ is a Lyapunov function with rectangular level curves. If $z^T A < 0$, then $V(x) = z^T|x|$ is a Lyapunov function which is linear in the positive orthant. Finally if $A^T P + PA \prec 0$ and $P \succ 0$, then $V(x) = x^T Px$ is a quadratic Lyapunov function for the system $\dot{x} = Ax$. See Fig. 1.

A discrete time counterpart will also be used:

Proposition 2. *For $B \in \mathbb{R}^{n \times n}_+$, the following statements are equivalent*:

(2.1) *The matrix B is Schur stable.*
(2.2) *There is a $\xi \in \mathbb{R}^n$ such that $\xi > 0$ and $B\xi < \xi$.*
(2.3) *There exists a $z \in \mathbb{R}^n$ such that $z > 0$ and $B^T z < z$.*
(2.4) *There is a diagonal $P \succ 0$ such that $B^T PB \prec P$.*
(2.5) *$(I - B)^{-1}$ exists and has nonnegative entries.*

Moreover, if $\xi = (\xi_1, \ldots, \xi_n)$ and $z = (z_1, \ldots, z_n)$ satisfy the conditions of (2.2) *and* (2.3) *respectively, then $P = \text{diag}(z_1/\xi_1, \ldots, z_n/\xi_n)$ satisfies the conditions of* (2.4).

To quantify control performance, it is useful to also discuss input–output gains. A remarkable feature of positive systems is that the input–output gain is determined by the static behaviour [17]:

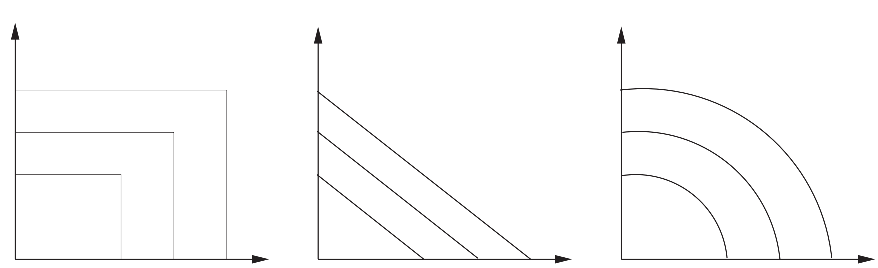

Fig. 1. Level curves of Lyapunov functions corresponding to the conditions (1.2), (1.3) and (1.4) in Proposition 1. See Remark 1.

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