$$R = \begin{pmatrix} 1 & 0 \\ -(x_1 - x_2)/2 & I_2 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & I_2/2 \\ -h^T(x_1 - x_2)[1 + (y + L^2)/2] & -h^T[I_2 + (x_1 - x_2)(x_1 - x_2)^T]/2 \end{pmatrix}$$

where I_2 is the identity matrix of dimension two. It follows that c is nonsingular iff $y \neq -L^2$ and h is not orthogonal to the bar. The feedback $\bar{u} = -c^{-1}a + c^{-1}\bar{v}$ produces $z^{(2)} = \bar{v}$ in closed loop, and so it is a solution of the relative-decoupling problem. Note that $n = 8$, $k^* = \tilde{k}^* = 2, \dim z = \tilde{\rho}(z) = 3$ and $\dim Y_{k^*-1} = 2$. Hence, from Theorem 2 part (iii), z is a relatively flat output. In particular, from Theorem 4 and Corollary 1, when the constraint $y(t) = 0$ is added, the given feedback is a decoupling and linearizing feedback law for the corresponding DAE.

8. Conclusions

The results of this paper may be useful for studying flatness and the dynamic decoupling problem for implicit systems. It is important to point out that our results show effective ways for computing the output rank and control laws for dynamic feedback linearization and/or decoupling of an implicit system Γ, without the need to transform Γ into an explicit system. In fact, note that the relative dynamic extension algorithm for affine systems relies only on sums, multiplications and matrix inversions.

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A Computation of the Static-Feedback of the kth Step of RDEA

Let

$$\bar{z}_k^{(k)} = \bar{a}(t, x_{k-1}) + \bar{b}(t, \tilde{x}_{k-1})\omega_k + \bar{c}(t, \tilde{x}_{k-1})\mu_{k-1}$$
$$\hat{z}_k^{(k)} = \hat{a}(t, \tilde{x}_{k-1}) + \hat{b}(t, \tilde{x}_{k-1})\omega_k + \hat{c}(t, \tilde{x}_{k-1})\mu_{k-1}$$

Up to a reordering of the components of z, we may assume that rank c = rank $\bar{c} = \tilde{\sigma}_k$ is locally constant. Up to a reordering of the components of μ_{k-1}, we may suppose that $\bar{c} = [\bar{c}_{11}\ \bar{c}_{12}]$, where $\bar{c}_{11}$ is locally nonsingular. Then define locally

$$\beta_k(t,\widetilde{x}_{k-1}) = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \bar{c}_{11}^{-1} & -\bar{c}_{11}^{-1}\bar{c}_{12} \\ 0 & I \end{pmatrix}$$

Let $\quad \bar{\alpha}_k(t,\widetilde{x}_{k-1}) + \widehat{\alpha}_k(t,\widetilde{x}_{k-1})\omega_k = \beta_k \begin{pmatrix} -\bar{a} - \bar{b}\omega_k \\ 0 \end{pmatrix}$

and let $\mu_{k-1} = \bar{\alpha}_k(t,\widetilde{x}_{k-1}) + \widehat{\alpha}_k(t,\widetilde{x}_{k-1})\omega_k + \beta_k(t,\widetilde{x}_{k-1})v_k$. Then it is easy to verify this choice of $(\bar{\alpha}_k, \widehat{\alpha}_k, \beta_k)$ is such that (18) holds.

B Proof of Lemma 2

Proof. Along this proof, we shall write $\omega = \omega_0$. By (19), it is clear that $\omega_k = \omega^{(k)}$ for $k = 0, 1, \ldots$. The following remark is instrumental for the proof:

Remark 9. Assume that $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$ is a state representation around ξ. Then by definition, $\Psi = \{t, \widetilde{x}_{k-1}, (\omega_k^{(j)}, \mu_{k-1}^{(j)} : j \in \mathbb{N})\}$ is a local coordinate chart around ξ. In particular, the differentials of the functions of Ψ are locally independent.

Let $(\widetilde{x}_{-1}, \widetilde{u}_{-1})$ be the state representation of system S with output $z^{(0)}$ defined by (16). In step $k-1$ of this algorithm ($k = 0, 1, 2, \ldots$) one has constructed a classical (local) state representation $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$, where $\widetilde{u}_{k-1} = (\omega_k, \mu_{k-1})$, with output $z^{(k)}$ defined on an open neighborhood U_{k-1} of $\xi \in S$. Assume that $\mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\omega_k, dz^{(k)}\}$ is nonsingular around ξ[16]. Note that we can give the following *geometric description of the step k* of RDEA:

(S1) Choose $\bar{z}_k$ (possibly among the components of z) by completing $\{dt, d\widetilde{x}_{k-1}, d\omega_k\}$ into a basis $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ for $\mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\omega_k, dz^{(k)}\}$.

(S2) Now choose $\widehat{\mu}_k$ (possibly among the components of $\widetilde{\mu}_{k-1}$) by completing $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ into a basis $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}, d\widehat{\mu}_k\}$ of $\mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\}$. According to the Section 2.1, this defines a local state feedback with new input[17] (ω_k, v_k), where $v_k = (\bar{z}_k^{(k)}, \widehat{\mu}_k)$. By construction, this state feedback has the property (18).

(S3) Define the new state representation $(\widetilde{x}_k, \widetilde{u}_k)$ by taking $\widetilde{x}_k = (\widetilde{x}_{k-1}, \omega_k, \bar{z}_k^{(k)})$, and $\widetilde{u}_k = (\dot{\omega}_k, \mu_k)$, where $\mu_k = (\bar{z}_k^{(k+1)}, \widehat{\mu}_k)$. This is an extension of the state of the form (19).

The proof of Lemma 2 relies on (S1), (S2), (S3). **(1 and 2)**. We show first that the state representation $(\widetilde{x}_k, \widetilde{u}_k)$ is classical, This property holds for $k = -1$. By induction, assume that it holds for $k-1$. Then from (S1), (S2) and (S3) we have $\mathrm{span}\{d\dot{\widetilde{x}}_k\} \subset \mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\bar{z}_k^{(k+1)}\} \subset \mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$. By (S1), (S2), (S3) notice that $d\,\widehat{z}_k^{(k+1)} \in \mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\dot{\bar{z}}_k^{(k)}\}$, and so, $\mathrm{span}\{dz^{(k+1)}\} \subset \mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$.

We show now 1 and 2 by induction. Since $\widetilde{x}_{-1} = x_{k^*-1}$, by part 1 of Lemma 1 it follows that $\mathrm{span}\{dt, d\widetilde{x}_{-1}\} = \mathcal{Y}_{k^*} = \mathcal{L}_{-1}$. By remark 2, from parts 1 and 8 of Lemma 1, and from the fact that $\mathrm{span}\{dz\} \subset \mathrm{span}\{dt, dx, du\} \subset \mathcal{Y}_{k^*} + \mathrm{span}\{du\}$ it follows that 1 and 2 are satisfied for $k = -1$. Assume that, in the step $k-1$ we have a local state representation $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$ satisfying 1 and 2. Choose a partition $z^{(k)} = (\bar{z}_k^{(k)}, \widehat{z}_k^{(k)})$ in a way that (S1) is satisfied and construct $\widehat{\mu}_k$ satisfying (S2). By 1 for $k-1$ and (S1) and from the fact that $\omega_k = \omega^{(k)}$, it follows that $\mathrm{span}\{dt, d\widetilde{x}_k\} = \mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\omega^{(k)}, d\bar{z}_k^{(k)}\} = \mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\omega^{(k)}, dz_k^{(k)}\} = \mathcal{L}_{k-1} + \mathrm{span}\{d\omega^{(k)}, dz^{(k)}\}$. From the fact that $\omega_0 = \omega = \bar{y}_{k^*}^{(k*)}$, by (14) and part 8 of Lemma 1, it follows that $\mathcal{L}_{k-1} + \mathrm{span}\{d\omega^{(k)}, dz^{(k)}\} = \mathcal{L}_k$, showing 1 for k.

We show now that if 2 holds for $k-1$, then $\mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\} = \mathcal{L}_{k+1} + \mathrm{span}\{du\}$, completing the induction. By (S1), (S2) and (S3) and from the fact that $\mathrm{span}\left\{d\bar{z}_k^{(k+1)}\right\} \subset \mathrm{span}\{dz^{(k)}\} \subset \mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$, it follows that $\mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\} = \mathrm{span}\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\} + \mathrm{span}\left\{d\omega^{(k+1)}, dz_k^{(k+1)}\right\}$. By the induction hypothesis, we have $\mathrm{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\} = \mathcal{L}_k + \mathrm{span}\{du\} + \mathrm{span}\{d\omega^{(k+1)}, dz_k^{(k+1)}\}$. By part 8 of Lemma 1 and the fact that $\omega = \bar{y}_{k^*}^{(k^*)}$, this shows 2 for k. **(3, 5, 6, 7)**. Note now that, since $\{dt, d\widetilde{x}_k\} = \{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ is a basis of $\mathcal{L}_k$ and $\{dt, d\widetilde{x}_{k-1}\}$ is a basis of $\mathcal{L}_{k-1}$, it follows that

$$\{d\omega_k\} \text{ is independent mod } \mathcal{L}_{k-1}. \tag{31}$$

In particular, $\{d\omega_k\}$ is also independent mod L_{k-1}. Since $\omega = \bar{y}_{k^*}^{(k^*)}$ and $\mathrm{card}\,\omega_k = \mathrm{card}\,\omega = \rho(y)$, by remark 1, we see that $\dim\mathcal{L}_k - \dim\mathcal{L}_{k-1} \geq \rho(y)$ and that $\dim L_k - \dim L_{k-1} \geq \rho(y)$. We show first that

$$\dim L_k(\nu) - \dim L_{k-1}(\nu) \geq \dim L_{k+1}(\nu) - \dim L_k(\nu), \text{ for every } \nu \in S_k \tag{32}$$

In fact, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset L_k$ are linearly dependent mod L_{k-1}, i.e., if $\alpha_0 dt + \sum_{i=1}^{s} \alpha_i \eta_i + \sum_{i=1}^{r} \sum_{j=0}^{k^*+k-1} \beta_{ij} dy_i^{(j)} + \sum_{i=1}^{p} \sum_{j=0}^{k-1} \gamma_{ij} dz_i^{(j)} = 0$, then differentiation in time gives $\dot{\alpha}_0 dt + \sum_{i=1}^{s} (\dot{\alpha}_i \eta_i + \alpha_i \dot{\eta}_i) + \sum_{i=1}^{r} \sum_{j=0}^{k^*+k-1} (\dot{\beta}_{ij} dy_i^{(j)} + \beta_{ij} dy_i^{(j+1)}) + \sum_{i=1}^{p} \sum_{j=0}^{k-1} (\dot{\gamma}_{ij} dz_i^{(j)} + \gamma_{ij} dz_i^{(j+1)}) = 0$. In other words, the 1-forms $\dot{\eta}_1, \ldots, \dot{\eta}_s$ are linearly dependent mod L_k. Let $\xi \in S_k$. From the nonsingularity of $L_j, \mathcal{L}_j, j = 0, \ldots, k$ in S_k, if $\dim L_k - \dim L_{k-1} = l + \rho(y)$ in $\xi \in S_k$, then by (31)

[16] It is easy to show that this is equivalent to the fact that the matrix $c_k(t, \widetilde{x}_{k-1})$ of (5) has constant rank around ξ.

[17] In fact, by construction we have that $\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\}$ and $\{dt, d\widetilde{x}_{k-1}, d\omega_k, dv_k\}$ are both local basis of the same codistribution.