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$\mathcal{H}_2$ optimization and fixed poles of sampled-data systems under colored noise

B.P. Lampe [a,*], E.N. Rosenwasser [b]

[a] Department of Computer Science and Electrical Engineering, University of Rostock, 18051 Rostock, Germany
[b] Department of Automatic Control, State University of Ocean Technology, 190008 St. Petersburg, Russia

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ABSTRACT

The paper considers the sampled-data control problem for multivariable continuous LTI processes when colored stationary stochastic disturbances act on their inputs. The design problem is to find a causal stabilizing controller, which ensures the minimal value of the mean output variance. Based on the parametric transfer matrix (PTM) concept, a constructive polynomial solution of that design problem is provided. A number of general properties of the optimal system is derived, which are important for practical applications. In particular, the existence of fixed poles is shown. A consequence of this fact is a principal performance limitation of the $\mathcal{H}_2$ optimal control loop.

1. Introduction

A fundamental task of direct design of sampled-data systems consists in the standard $\mathcal{H}_2$−optimization problem [5,2,15,12,24,6,11,17,1,23,20,21] which leads to the construction of a stabilizing discrete controller that ensures the minimal mean variance of the output of the closed sampled-data system under vector white noise acting on the input. Existing methods for the solution of the standard sampled-data $\mathcal{H}_2$−optimization problem can be classified into two groups. The design methods of the first group are connected with the solution of an associated matrix Riccati equation. The best known instances of this group are the lifting method [2,15,24,6] and the FR-operator method [11]. Also the hybrid state space method [12], and the indirect discretization method developed in [17,1], belong to this class.

As an alternative to the above methods we can consider the frequency polynomial approach associated with the application of the parametric transfer matrix (PTM), [20,21]. This approach allows to apply the idea of the Wiener–Hopf method in the form [25]. As shown in [20,21], the application of the PTM method results in the standard $\mathcal{H}_2$ problem for the optimal system, where its characteristic polynomial possesses reduced degree in comparison with applying the Riccati equation method. Moreover, for the standard $\mathcal{H}_2$ problem, the PTM method discloses a number of up to now unknown properties of the optimal system. In particular, we found that among the poles of the continuous LTI process, there exists in dependence of the structure of the SD system a set $\mathcal{M}_f$ of fixating poles, which are characterized by the following properties:

(a) When among the set of fixating poles, there are poles on the imaginary axis, then the $\mathcal{H}_2$−optimization problem does not possess a solution in the set of causal stabilizing controllers.
(b) Let the set of fixating poles be composed of the numbers $s_1,\dots,s_q$, where $\operatorname{Re} s_i<0$ for $i=1,\dots,\eta$, and $\operatorname{Re} s_i>0$ for $i=\eta+1,\dots,q$. Then the set of poles of the optimal system contains the numbers $\zeta_i=e^{-s_iT}$, $i=1,\dots,\eta$, and $\zeta_i=e^{s_iT}$, $i=\eta+1,\dots,q$. Further on those poles are called fixed ones. Previously, the existence of fixed poles at the $\mathcal{H}_2$ optimization problem for continuous or discrete LTI systems was described in [3,4].

The present paper extends the PTM method for the solution of the generalized $\mathcal{H}_2$−optimization problem, when stationary colored noise acts on the input of the SD system. In principle, this problem can be solved on basis of the Riccati equation method by extending the state space model of the continuous process. However, in the given case the application of the PTM method results, as for the standard $\mathcal{H}_2$ problem, in a reduced order optimal system, and the offered approach grants to detect a number of general properties of the optimal system, which are unknown up to now, but important for practical applications in monitoring and control [10]. In particular, the following properties should be quoted:

(1) In the generalized $\mathcal{H}_2$ problem, analogously to the standard problem, there exists a set of fixating poles of the LTI process, which depend on the structure of the process, the attack point of the excitation and the location of the SD system output.

* Corresponding author.
E-mail address: bernhard.lampe@uni-rostock.de (B.P. Lampe).

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(2) The set of fixed poles for the standard and the generalized $\mathcal{H}_2$ optimization problem coincide.
(3) Those poles determined by the properties of the coloring filter of the input signal are formally configured by the factorization, but they do not belong to the fixed poles, and under some assumptions, which are satisfied practically always, they will be canceled during the construction procedure of the optimal controller.

The above-named properties of the optimal system have to be regarded at the solution of practical problems. Here the following should be mentioned:

(i) The fact that a set of fixed poles exists, essentially restricts the performance of $\mathcal{H}_2$ optimal systems. When the set of fixating poles contains poles located close to the imaginary axis, then the corresponding fixed poles are located close to the unit circle, i.e. the system is close to the stability bound. Therefore, an appropriate SD control system design on basis of the $\mathcal{H}_2$ optimization methods needs to construct the set of fixating poles of the continuous process and to study its properties.
(ii) As known, the $\mathcal{H}_\infty$ optimization is a limit case of a generalized $\mathcal{H}_2$ optimization problem. Therefore, the fact that the set of fixed poles exists and is independent on the spectrum of the input signal in the generalized $\mathcal{H}_2$ optimization problem is transferred also to the $\mathcal{H}_\infty$ optimal system. For SISO systems this statement is strongly proven in [18].

The paper is organized as follows. Section 2 provides the description of the system and the formulation of the problem. Section 3 gives some preliminaries on polynomial and rational matrices. Most of these facts are known, but they are presented in a form which is used in the further considerations.

In Sections 4 we consider fundamental properties of parametric discrete models of continuous processes. Section 5 provides the polynomial procedure for constructing the transfer matrix of the optimal controller. In Section 6 we consider general properties of the optimal system, and we proof the reduction of the poles of the forming filter. Finally, in Section 7 we demonstrate the application of the provided method at hand of an example.

2. System description and problem

(1) The paper considers the sampled-data control problem for the continuous process described by the state space equation

$$\frac{dv(t)}{dt} = Av(t) + B_1x(t) + B_2u(t) \tag{1}$$

and the output equation

$$y(t) = C_2v(t), \tag{2}$$

where $y(t)$, $v(t)$, $u(t)$, $x(t)$ are output, state, control and disturbance vectors of the dimensions $n\times 1$, $p\times 1$, $m\times 1$, $l\times 1$, respectively, and A, B_1, B_2, C_2 are constant matrices of appropriate size.
(2) Assume that the process (1), (2) is controlled by the digital computer described by a linearized model

$$\xi_k = y(kT), \quad (k = 0, \pm 1, \ldots), \tag{3}$$

$$\alpha_0\psi_k + \ldots + \alpha_q\psi_{k-q} = \beta_0\xi_k + \ldots + \beta_q\xi_{k-q}, \tag{4}$$

$$u(t) = h(t-kT)\psi_k, \quad kT < t < (k+1)T. \tag{5}$$

In Eqs. (3)–(5), the quantity $T > 0$ is the sampling period, α_i, β_i are constant $m\times m$ and $m\times n$ matrices, respectively. Moreover, in (5), $h(t)$ is a piecewise smooth function defining the form of the control impulses [23,20].
(3) Eq. (4) is called *discrete controller equation*. Introduce the backward shift operator $\zeta = e^{-sT}$ [1], then the control algorithm can be written in the form

$$\alpha(\zeta)\psi_k = \beta(\zeta)\xi_k,$$

where $\alpha(\zeta)$, $\beta(\zeta)$ are polynomial matrices of the form

$$\alpha(\zeta) = \alpha_0 + \alpha_1\zeta + \cdots + \alpha_q\zeta^q, \quad \beta(\zeta) = \beta_0 + \beta_1\zeta + \cdots + \beta_q\zeta^q.$$

Further on, the polynomial pair $(\alpha(\zeta), \beta(\zeta))$ is called discrete controller, or shortly controller. The controller $(\alpha(\zeta), \beta(\zeta))$ is called *causal*, when

$$\det \alpha_0 \neq 0.$$

As known, in practice only causal controllers could be realized in real time. For a causal controller, the rational matrix

$$W_d(\zeta) = \alpha^{-1}(\zeta)\beta(\zeta)$$

is always defined, which is called its *transfer matrix*.
(4) Further on, the differential-difference equation system (1)–(5) is called the system $\mathcal{S}$. As in [6,8,21], the system $\mathcal{S}$ is called *stable*, when for $x(t) = O_{l1}$, where O_{nm} is the $n\times m$ zero matrix, and arbitrary initial conditions for $t > 0$, $k > 0$ an estimate

$$\|v(t)\| < C_v e^{-\delta t}, \ \|u(t)\| < C_u e^{-\delta t}, \ \|\psi_k\| < C_\psi e^{-\delta kT}$$

takes place, where $\|\cdot\|$ is the Euclidean norm and C_v, C_u, C_ψ, δ are positive constants, and δ can be chosen independently of the initial conditions.
(5) Below we consider the $k\times 1$ vector

$$z(t) = C_1v(t) + Du(t) \tag{6}$$

as output of the system $\mathcal{S}$, where C_1, D are constant matrices. Let the system $\mathcal{S}$ be stable, and the input signal should be the transition result of a $d\times 1$ vector of centered white noise through a stable continuous input filter with transfer matrix $\Phi(s)$. Then, as follows from [20,21], after finishing the transient processes, the output $z(t)$ is a centered periodically non-stationary process denoted by $z_\infty(t)$, and its variance $d_z(t) = d_z(t+T)$ is determined by the formula

$$d_z(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[W'_{zx}(-s,t)W_{zx}(s,t)\Phi(s)\Phi'(-s)]\, ds, \tag{7}$$

where $j = \sqrt{-1}$, tr means the trace of a matrix, and the prime indicates the transposition operation. In formula (7), $W_{zx}(s,t) = W_{zx}(s,t+T)$ is the parametric transfer matrix (PTM) of the system $\mathcal{S}$ from input $x(t)$ to output $z(t)$. Formula (7) can be found in [21], or may be derived directly from relations (88) and (89) in Section 5. The mean variance of the output $z_\infty(t)$ over one period $\bar{d}_z$ is determined by

$$\bar{d}_z = \frac{1}{T}\int_0^T d_z(t)\, dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[A_{zx}(s)\Phi(s)\Phi'(-s)]\, ds, \tag{8}$$

where

$$A_{zx}(s) = \frac{1}{T}\int_0^T W'_{zx}(-s,t)W_{zx}(s,t)\, dt. \tag{9}$$

(6) Using the above concepts, the following generalized $\mathcal{H}_2$–optimization problem can be formulated: **Generalized $\mathcal{H}_2$–optimization problem:** Let the matrices A, B_1, B_2, C_1, C_2, D, the sampling period T, the function $h(t)$ and the transfer matrix of the input filter $\Phi(s)$ be given. Moreover, let the system $\mathcal{S}$ be stabilizable. Find a causal stabilizing controller $(\alpha^0(\zeta), \beta^0(\zeta))$, for which the mean variance $\bar{d}_z$ takes the minimal value.