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A separation principle for linear impulsive systems

Imen Ellouze [a], Mohamed-Ali Hammami [a], Jean-Claude Vivalda [b,c,d,*]

[a] Université de Sfax, Département de Mathématiques, Route de Soukra, BP1171, Sfax 3000, Tunisia
[b] Université de Lorraine, IECL, UMR 7502, Vandoeuvre-lès-Nancy F-54506, France
[c] CNRS, IECL, UMR 7502, Vandoeuvre-lès-Nancy F-54506, France
[d] Inria, Villers-lès-Nancy F-54600, France

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ABSTRACT

In this paper, we deal with the problem of the stabilization and the estimation of the state for a class of linear impulsive systems. We show that, under the classical Kalman's conditions of controllability and observability it is possible to design an observer based controller for the linear *impulsive* system.

1. Introduction

Compared to classical continuous or discrete systems, impulsive systems can involve instantaneous and discontinuous changes at various time instants. Differential equations involving impulse effects occur in many applications: population dynamics in relation to impulsive vaccination [11], population ecology [10], drug distribution in the human body [2], management of renewable resources, etc.

Fundamental problems of automatic control theory, such as observability, reachability and controllability, have been widely investigated for different types of impulsive systems [1,4,14,15]. Guan et al. in [1] developed controllability and observability results for linear impulsive systems where the control is available for the continuous-time dynamics only at impact time and the impulsive effects are limited to scalings of the state. For the same type of linear impulsive systems, Xie and Wang in [14] used a geometric framework to generalize the results of Guan et al. [1]. Medina and Lawrence [5] presented a geometric characterization of the reachable and unobservable subspaces for a more general class of linear impulsive systems. Medina and Lawrence [7] obtained the result of state feedback stabilization for a class of linear impulsive systems with arbitrarily spaced impulse times and possibly singular state transition matrices. In contrast to many well-established observers, which normally estimate the system state in an asymptotic fashion, Raff et al. in [8] proposed an impulsive observer with predetermined finite convergence time for linear continuous-time systems, such that the observer state is updated at a definite time instant. In [13], the authors present a characterization of observability and an observer design method for switched linear systems with impulses. In particular, they give a necessary and sufficient condition for observability.

In this paper we focus on the construction of observers for linear impulsive systems with the objective of stabilizing the state of the system about the origin thanks to a dynamic output feedback. On this subject, we have to mention the paper [6] in which the authors built an observer for linear impulsive systems with a *discrete-time* observation function; under the assumption of strong observability property, they show that their observer yields an uniformly exponentially stable dynamics for the error estimation. In this paper, we treat the case of *continuous-time* observation function. Our point of view is to regard the impulsions as perturbations, under the hypothesis that these impulsions do not occur too often, we deal with the problem of the stabilization and the design of a Luenberger like observer. We also show a separation principle. The paper is organized as follows. In Section 2, we state the key lemma on which our results will be based. In Section 3, we show, how under the classical assumptions of controllability and observability of the pairs (A,B) and (A,C), we can construct a stabilizing feedback law and an observer. In Section 4, we give a separation principle. Finally, in the last section, we construct an observer based in the impulsive part.

* Corresponding author at: Université de Lorraine, IECL, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France.
E-mail address: Jean-Claude.Vivalda@inria.fr (J.-C. Vivalda).

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2. The systems under consideration. The Key lemma

The class of systems under consideration is described by the following equations:

$$\begin{cases} \dot{x}(t) = Ax(t) - Bu(t) & t \in [t_{k-1} t_k) \\ x(t_k) = Dx(t_k^-) \\ y(t) = Cx(t) \end{cases} \quad (1a–c)$$

where the state x belongs to $\mathbf{R}^n$, the control u is in $\mathbf{R}^m$, y is in $\mathbf{R}^p$ and the matrices A, B, D and C have appropriate dimensions. Moreover, $x(t_k^-)$ denotes the limit $\lim_{\substack{t \to t_k \\ t < t_k}} x(t)$; the solution of (1) is thus piecewise continuous and continuous to the right. We assume that the sequence of impulse times $t_0, t_1, \ldots$ is increasing and, for every $k \geq 1$, we denote by δ_k the difference $\delta_k = t_k - t_{k-1}$. We assume also that Kalman's conditions are satisfied for the pairs (A,B) and (A^T, C^T). Under these assumptions, it is well known that, given an arbitrary positive number r, there exists a gain matrix K such that the spectrum of $A_K \triangleq A - BK$ is located in the half plane $H_r = \{z \in \mathbf{C} \mid \mathrm{Re}(z) < -r\}$. In other words, given $r > 0$, there exists a gain matrix K such that $\| e^{tA_K} \| \leq c(r,K) e^{-rt}$ where $c(r,K)$ is a constant (and $\| \cdot \|$ is any multiplicative norm on $\mathbf{R}^{n \times n}$). The constant $c(r,K)$ depends on r and K in a way that has been analyzed by Sussmann and Kokotovic in [12]. Denoting by $c(r)$ the *infinum* of the numbers $c(r,K)$ taken over all the gain matrices K such that the spectrum of A_K is included in H_r, these authors have proved that

$$c(r) \leq ar^{\nu}$$

where a is a constant and ν is the largest controllability index of the pair (A,B). The following lemma is a direct consequence of this result. Hereafter, we denote by S a compact set included in $H_0 = \{z \in \mathbf{C} \mid \mathrm{Re}(z) < 0\}$ and which is symmetric about the Ox-axis. Given a real number $\alpha > 0$, we denote by αS the set $\alpha S = \{\alpha z \mid z \in S\}$. A set $\{\lambda_1, \ldots, \lambda_n\}$ of complex numbers is called a *spectrum* if the coefficients of the polynomial $(X - \lambda_1) \cdots (X - \lambda_n)$ are all real numbers.

Lemma 2.1. *Take a set $S \subset H_0$ as described above and let $r > 0$ be such that $\sup\{\mathrm{Re}(z) \mid z \in S\} < -r$. Then there exists a constant $a > 0$ (depending on S and on the pair (A,B)) such that, for every $\alpha \geq 1$ and every spectrum $\{\lambda_1, \ldots, \lambda_n\}$ included in $(\alpha S)^n \subset \mathbf{C}^n$, there exists a gain matrix K such that the eigenvalues of A_K are the λ_i's and A_K satisfies the following inequality:*

$$\| e^{tA_K} \| \leq a(\alpha r)^{\nu - 1} e^{-\alpha r t}, \quad \text{for every } t \geq 0$$

where ν denotes the largest controllability index of the pair (A,B).

Proof. The proof can be easily derived from the results shown in [12] (especially Theorem 8.1). □

Corollary 2.2. *Denote by S a compact set included in H_0, which is symmetric about the Ox-axis and let b, d and τ be three given positive numbers. Then there exists a constant $\alpha \geq 1$ (depending on b, d and τ) such that for any spectrum $\Lambda \in (\alpha S)^n \subset \mathbf{C}^n$, there exists a gain matrix K such that the spectrum of A_K is equal to Λ; moreover A_K satisfies the inequality*

$$\| e^{tA_K} \| \leq \mu e^{-b' t} \quad \text{for every } t \geq 0,$$

where $b' > b$ and $\mu > 0$ is a constant such that

$$d\mu e^{-b'\tau/2} \leq 1.$$

Proof. Let $r > 0$ be such that $\sup\{\mathrm{Re}(z) \mid z \in S\} < -r$ and take $\alpha \geq 1$ such that $\alpha r > b$ (so $\alpha S \subset H_b$). Let $\Lambda \in (\alpha S)^n \subset \mathbf{C}^n$ be a spectrum, then from Lemma 2.1, there exist a constant a and a gain matrix K such that the spectrum of A_K is equal to Λ; moreover A_K satisfies the inequality

$$\| e^{tA_K} \| \leq a(\alpha r)^{\nu - 1} e^{-\alpha r t}.$$

As $\lim_{\alpha \to +\infty} d a(\alpha r)^{\nu - 1} e^{-\alpha r \tau/2} = 0$, we have

$$da(\alpha r)^{\nu - 1} e^{-\alpha r \tau/2} \leq 1, \quad (2)$$

for every α large enough. Let $\mu = a(\alpha r)^{\nu - 1}$, then we have $\| e^{tA_K} \| \leq \mu e^{-b' t}$ and $d\mu e^{-b'\tau/2} \leq 1$ where $b' = \alpha r$. □

Remark. If the pair (A,C) is observable, from this result, we deduce easily the existence of a gain matrix L such that $\| e^{t(A-LC)} \| \leq \mu e^{-b't}$ for every $t \geq 0$ with $d\mu e^{-b'\tau/2} \leq 1$: just apply the lemma to the pair (A^T, C^T) which is controllable and notice that the norm of a matrix is equal to the one of its transposed.

3. Stabilization and estimation of the state of linear impulsive systems

We shall construct now a linear feedback law which makes the origin of system (1) exponentially stable.

Theorem 3.1. *Assume that the pair (A,B) is controllable and that the sequence $(\delta_k)_{k \geq 1}$ is bounded from below, then one can choose a gain matrix K in such a way that the closed-loop system*

$$\begin{cases} \dot{x}(t) = (A - BK)x(t) & t \in [t_{k-1}, t_k) \\ x(t_k) = Dx(t_k^-) \end{cases} \quad (3)$$

is globally exponentially stable about the origin. Moreover the speed of convergence can be made arbitrarily large according to the choice of matrix K.

Proof. The sequence $(\delta_k)_{k \geq 1}$ being bounded from below, there exists $\tau > 0$ such that $\delta_k > \tau$ for every $k \geq 1$. Thanks to the above corollary, we know that we can choose $b > 0$, arbitrarily large, and a gain matrix K such that

$$\| e^{tA_K} \| \leq \mu e^{-bt} \quad \text{and} \quad \mu e^{-b\tau/2} \| D \| \leq 1.$$

We shall show that, with this feedback law, the linear impulsive system (3) is exponentially stable about the origin.

The state transition matrix of (3) is given by

$$\phi(t, t_0) = e^{(t - t_k)A_K} D e^{\delta_k A_K} \cdots D e^{\delta_1 A_K}, \quad t \in [t_k, t_{k+1}),$$

so we have

$$\begin{aligned} \| \phi(t, t_0) \| &\leq \|D\|^k \| e^{(t - t_k)A_K} \| \, \| e^{\delta_k A_K} \| \ldots \| e^{\delta_1 A_K} \| \\ &\leq \|D\|^k \mu^{k+1} e^{-b(t - t_k)} e^{-b\delta_k} \ldots e^{-b\delta_1} \\ &= \|D\|^k \mu^{k+1} e^{-b(t - t_k)} e^{-b(t_k - t_0)} \\ &\leq \mu (\| D \| \mu e^{-b\tau/2})^k e^{-b(t - t_k)} e^{-b/2(t_k - t_0)} \quad \text{because } \delta_i \geq \tau \\ &\leq \mu e^{-b(t + t_k/2 - t_0/2)} \quad \text{because } \| D \| \mu e^{-b\tau/2} \leq 1 \\ &\leq \mu e^{-b/2(t - t_0)}. \end{aligned}$$

This inequality proves the claim of the global asymptotic stability of (3) (and even the exponential stability). Moreover, as b can be chosen arbitrarily, the speed of convergence can be made arbitrarily large. □

We consider now the following auxiliary dynamical system:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) - Bu(t) - L(C\hat{x}(t) - y(t)), & t \in [t_{k-1}, t_k) \\ \hat{x}(t_k^+) = D\hat{x}(t_k^-) \end{cases} \quad (4)$$

by arguing in exactly the same way as in Theorem 3.1, we prove that the gain matrix L can be chosen in such a way that system (4) is an exponential observer for system (1).

Theorem 3.2. *Assume that the pair (A,C) is observable and that the sequence $(\delta_k)_{k \geq 1}$ is bounded from below, then one can choose the gain matrix L in such a way that system (4) is an exponential*