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High-gain observers for nonlinear systems with trajectories close to unobservability

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ABSTRACT

High-gain observers can be easily designed for nonlinear systems and they present very good convergence properties, when the observability mapping of the system is injective in the whole. If the observability mapping is injective in the whole, the system can be transformed into a normal form, from which the estimation of the successive output derivatives can be easily carried out. If the observability mapping is not injective in the whole and the state trajectory of the system passes close to some set where the property of observability is lost (about which the observability mapping is not injective), the resulting high-gain observers may be subject to significant numerical errors. The objective of the paper is to propose an improvement of the classical procedure for the high-gain observer design, so that the nonlinear system is transformed into a modified normal form, through a transformation that is obtained from the observability mapping, by removing singularities belonging to a certain class. Hence, a new high-gain structure, containing two sets of small parameters, is proposed for designing a high-gain observer for this modified normal form. The new observer is applied to a Bullard dynamo, to show that the new observer obtains good state estimates even for trajectories very close to unobservability, for which the classical high-gain observer is affected by significant numerical errors.

1. Introduction

The state observation problem for (possibly, time-varying) linear systems was introduced, with the modern terminology, in [34] and solved in the deterministic case in [43,44]. A survey about state observers for linear systems can be found in [57]. A geometric characterization of the observability for linear systems is given in [71] (see also [6], in the case of linear systems, and [32,37,54,59], in the case of nonlinear systems).

Many definitions of observability in the case of nonlinear systems have been given in [28,53,60–62], including the definition of observability independent of the input [20], which has been applied in the high-gain observer design.

The problem of obtaining state estimates for nonlinear systems is a very challenging field of research, and has been continuously attracting the attention of many researchers in the area of control theory in the past few decades, both for its intrinsic value and for the possible use of the state estimates for control purposes. Without any claim to be complete, some of the main contributions (not adopting high-gain structure) can be found in [2,4,7,12,13,35, 39–41,47–49,58,67–69]; such works, and references therein, often include discussions on different concepts and definitions related to the problem of state-estimation.

Starting from similar ideas adopted in the analysis of singularly perturbed systems [38], the use of high-gain in the observer design for nonlinear systems has been introduced in the late 1980s and early 1990s, from theoretical [9,15,16,64,65] and applicative [21,50–52] points of view; as an application of high-gain observer in control applications, in [63], Teel and Praly combined results from Tornambe [66] and Esfandiari and Khalil [16] to give the first non-local separation principle for nonlinear systems (see [32]).

Among the more recent contributions, without any claim of completeness, one can mention [1,3,5,8,11,14,18,19,26,27,30,31,42,55, 70,72]. Surveys about high-gain observers can be found in [22,36].

In the classical high-gain observer design, the considered nonlinear system is diffeomorphic to a normal form, which is obtained by using as state transformation the observability mapping; this normal form is observable in the whole, if the observability mapping is injective in the whole. A point, about which the observability mapping is not injective, is said to be singular. About singular points both the inverse of the observability mapping and the normal form are not defined. If the state trajectory of the system passes very close to some singular point, the computation of the inverse of the

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observability mapping may be affected by numerical errors and the normal form may contain terms that tend to be unbounded when the trajectory passes close to the singularity. The objective of the paper is to propose an improvement of this classical procedure, so that the nonlinear system is transformed into a modified normal form, through a transformation that is obtained from the observability mapping, by removing singularities belonging to a certain class: in this way, both the transformation does not present singularities and the modified normal form does not contain unbounded terms in proximity of the singularities of the observability mapping; this is paid by the fact that such a modified normal form may not be observable in the whole. Hence, a new high-gain structure, containing two sets of small parameters, is proposed for designing a high-gain observer for this modified normal form, allowing the state trajectory of the system to pass close to singular points, but not over them. The results are very good, as proved in a simulation test, where the new observer is applied to a Bullard dynamo, showing that the new observer obtains good state estimates even for trajectories very close to unobservability.

The organization of the paper is as follows: after a first brief review of the singularities that can be presented by the observability mapping, in Section 3 a physical system is described, the Bullard dynamo, which is locally observable in a set that is dense in $\mathbb{R}^2$, and loses its observability along a line in the phase plane. For trajectories approaching such a line (which are not rare) the traditional high-gain observers fail in obtaining the estimate of the non-measured variable. In Section 4, a new approach to high-gain observer design is proposed, consisting in a change of coordinates that allows us, under some technical assumptions, to rewrite the system in a modified normal form, and to "remove" singularities related to the loss of local observability; based on such a normal form a new design of the output injection matrix gain is given, which uses two small parameters, whose choice allows us to obtain practical stability of the estimation error dynamics, and, in lucky cases, exponential stability. Finally, in Section 5, the new observer is applied to the Bullard dynamo, showing a dramatic improvement of performance with respect to traditional high-gain observer.

2. Singularities of the observability mapping

The goal of this section is to review some basic definitions of observability for nonlinear systems, with the aim of characterizing the main difficulty to be addressed in this paper: the possible loss of observability due to singularities of the observability mapping, occurring just at some points of the state space. Several simplifying choices are made, to concentrate on the description of the loss of observability that motivates the observer proposed later in the paper. Consider a system

$$\dot{\xi} = f(\xi), \tag{1a}$$

$$y = h(\xi), \tag{1b}$$

where $\xi \in \mathbb{R}^n$ and f and h are analytic in $\mathcal{U} \subset \mathbb{R}^n$. For the sake of simplicity, assume in this section that $y \in \mathbb{R}$ (this assumption will be removed in the following sections). The *observability mapping* is given by

$$H(\xi) = \begin{bmatrix} h(\xi) \\ L_f h(\xi) \\ \vdots \\ L_f^{n-1} h(\xi) \end{bmatrix},$$

where, for any scalar function $\alpha(\xi)$, $L_f\alpha$ is the directional derivative of α along f, and $L_f^{i+1}\alpha = L_f(L_f^i\alpha)$.

Remark 1. As explained in [22], the choice made here of limiting to $n-1$ the order of derivative of h in defining the observability mapping is somewhat restrictive, since it may be needed to derive up to the order $2n-1$ to obtain an injective mapping. The approach proposed here could also be extended to the cases when it is needed to derive up to order $2n-1$. Nevertheless, since the focus of this paper is on how to behave when there are singularities, of the kind described below, for simplicity the definition above has been preferred.

System (1) is *locally differentially observable* (briefly, locally observable) at $\xi_0 \in \mathcal{U}$ if there exists a neighborhood $\mathcal{B}_{\xi_0}$ of ξ_0 such that $H(\xi)$ is an injective mapping on $\mathcal{B}_{\xi_0}$; system (1) is *locally differentially observable* (briefly, locally observable) in $\mathcal{U}$ if it is locally differentially observable at each $\xi_0 \in \mathcal{U}$. If $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0}) = n$, then system (1) is locally observable at $\xi_0 \in \mathcal{U}$, but condition $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0}) < n$ need not imply that system (1) is not locally observable at $\xi_0 \in \mathcal{U}$. Some functions $\alpha_1(\xi), \ldots, \alpha_m(\xi)$, being analytic on $\mathcal{U}$, are *functionally dependent* on $\mathcal{U}$ if for each $\xi_0 \in \mathcal{U}$ there exists a function F_{ξ_0}, not identically zero, such that $F_{\xi_0}(\alpha_1(\xi), \ldots, \alpha_m(\xi)) = 0$ for all $\xi \in \mathcal{B}_{\xi_0}$, where $\mathcal{B}_{\xi_0}$ is a neighborhood of ξ_0, and *functionally independent* if for each $\xi_0 \in \mathcal{U}$ there exists no function F_{ξ_0} such that $F_{\xi_0}(\alpha_1(\xi), \ldots, \alpha_m(\xi)) = 0$ for all ξ belonging to some $\mathcal{B}_{\xi_0}$. By Theorem 2.16 of [56] (see also Theorem 1.1 of [45]), functions $\alpha_1(\xi), \ldots, \alpha_m(\xi)$ are functionally independent (respectively, dependent) of $\mathcal{U}$ if and only if the rank of $\partial\alpha/\partial\xi$, where $\alpha = [\alpha_1, \ldots, \alpha_m]^\top$, is (respectively, is not) full over the field of meromorphic functions. It is worth pointing out that functional dependence and functional independence exhaust the range of possibilities only in the analytic case (for more details, see [56]), whereas the situation is more complicated in the non-analytic case.

Since f and h are analytic on $\mathcal{U}$, the following three cases are possible.

(*Case* 1) The rank of $\partial H/\partial\xi|_{\xi=\xi_0}$ is full for each $\xi_0 \in \mathcal{U}$, whence system (1) is locally observable in $\mathcal{U}$. In this case, the functions $h, L_f h, \ldots, L_f^{n-1}h$ are functionally independent and the functions $h, L_f h, \ldots, L_f^{n-1}h, L_f^n h$ are functionally dependent; therefore, for each $\xi_0 \in \mathcal{U}$, there exists a function ϕ_{ξ_0} such that $\phi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^n h(\xi)) = 0$, for all $\xi \in \mathcal{B}_{\xi_0}$, where $\mathcal{B}_{\xi_0}$ is a neighborhood of ξ_0. Since $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0}) = n$, equation $\phi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^n h(\xi)) = 0$ can be locally rendered explicit with respect to $L_f^n h$, whence there exists a function φ_{ξ_0} such that $L_f^n h(\xi) = \varphi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^{n-1}h(\xi))$, for all $\xi \in \mathcal{B}_{\xi_0}$.

(*Case* 2) The entries of $H(\xi)$ are functionally independent (i.e., the rank of $\partial H/\partial\xi$ is full over the field of meromorphic functions) but there may exist some point $\xi_0 \in \mathcal{U}$ such that $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0})$ is not full, whence system (1) is locally observable for almost all $\xi_0 \in \mathcal{U}$ because the set of points $\xi_0 \in \mathcal{U}$, for which the rank of $\partial H/\partial\xi|_{\xi=\xi_0}$ is not full, is nowhere dense in $\mathcal{U}$ (i.e., its complement is dense in $\mathcal{U}$). Also in this case, the functions $h, L_f h, \ldots, L_f^{n-1}h$ are functionally independent and the functions $h, L_f h, \ldots, L_f^{n-1}h, L_f^n h$ are functionally dependent; therefore, for each $\xi_0 \in \mathcal{U}$, there exists a function ϕ_{ξ_0} such that $\phi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^{n-1}h(\xi), L_f^n h(\xi)) = 0$, for all $\xi \in \mathcal{B}_{\xi_0}$, where $\mathcal{B}_{\xi_0}$ is a neighborhood of ξ_0. If $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0}) = n$, then equation $\phi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^n h(\xi)) = 0$ can be locally rendered explicit with respect to $L_f^n h$ about ξ_0, but if $\text{rank}(\partial H/\partial\xi|_{\xi=\xi_0}) < n$, then it could happen that equation $\phi_{\xi_0}(h(\xi), L_f h(\xi), \ldots, L_f^{n-1}h(\xi), L_f^n h(\xi)) = 0$ is not locally explicitable with respect to $L_f^n h$ about ξ_0.